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INDEFINITE QUADRATIC FORMS IN MANY VARIABLES

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In this paper we investigate the positive and the nonnegative inhomogeneous spectra for indefinite binary quadratic forms of rank at least 21. (Recent work of Margulis should allow the restriction $n \geqslant 21$ to be reduced.) For the positive spectra we prove that the sharpest constant depends only on the congruence class (mod 8) of the signature of the form; all extreme forms and grids are also obtained. In addition we bound the best possible constant for all nonnegative inhomogeneous spectra.

1. INTRODUCTION

Let $f(X) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j$ be a nonsingular quadratic form with real coefficients.

Using the notation in Watson[31], we let

$$A = \begin{pmatrix} 2a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & 2a_{22} & \cdots & a_{2n} \\ a_{1n} & a_{2n} & \cdots & 2a_{nn} \end{pmatrix}$$

be called the matrix associated with f and set $D = \det f$ to equal the determinant of A. Also, the discriminant of f is defined to be

$$d(f) = (-1)^{[n/2]} \cdot \begin{cases} \det f & \text{if } n \text{ is even} \\ \dfrac{\det f}{2} & \text{if } n \text{ is odd} \end{cases} \qquad \ldots(1)$$

In Watson[31] (p. 3), it is shown that the discriminant of any integral quadratic form is an integer; the nonsingularity of f implies that each of $\det f$ and $d(f)$ is nonzero. Further, in Watson[31] (p. 3), it is shown that, if $f_1 + f_2$ is a disjoint sum, then

$$d(f_1 + f_2) = \begin{cases} d(f_1)\, d(f_2) & \text{if } n_1 n_2 \text{ is even} \\ -4d(f_1)\, d(f_2) & \text{if } n_1 n_2 \text{ is odd.} \end{cases} \qquad \ldots(2)$$

Also, $d(f)$ is an integer and by Watson[31] (p. 21),

$$d(f) \equiv 0, 1 \pmod 4, \text{ if } n \text{ is even.} \qquad \ldots(3)$$

For any $V \in \mathbb{R}^n$, we define

$$P^+(f, \mathrm{V}) = \inf \{f(\mathrm{X} + \mathrm{V}) > 0 : \mathrm{X} \text{ is an integral point}\};$$

$$P(f, \mathrm{V}) = \inf \{f(\mathrm{X} + \mathrm{V}) > 0 : \mathrm{X} \text{ is an integral point}\}$$

where for the special case $\mathrm{V} = \mathbf{0}$ we restrict to $\mathrm{X} \neq \mathbf{0}$. We define

$$P_I^{\,+}(f) = \sup \{P^+(f, \mathrm{V}) : \mathrm{V} \in \mathbb{R}^n\};$$

$$P_I(f) = \sup \{P(f, \mathrm{V}) : \mathrm{V} \in \mathbb{R}^n\};$$

which may be called, respectively, the positive and nonnegative nonhomogeneous minima of the form f. For fixed $n > 1$, $| s | \leqslant n$, we consider the sets

$$\{P_I^{\,+}(f)^n / | D | : f \text{ indefinite of rank } n \text{ and signature } s\};$$

$$\{P_I(f)^n / | D | : f \text{ indefinite of rank } n \text{ and signature } s\}$$

which we shall call, respectively, the positive and nonnegative inhomogeneous spectra.

Bambah *et al.*[2,3] have considered the positive spectrum. They show that, for $n > 2$ and $s = 0, 1, 2, 3$; $n \geqslant 7$ and $s = -1$,

$$P_I^{\,+}(f)^n / | D | \leqslant (| s | + 1)^{-1}.$$

They also prove that this bound is best possible for each of these values of n, s. Hence, this extends the work of many authors, among them : Davenport and Heilbronn[11], Blaney[6], Barnes[4], Dumir[14,15], Hans-Gill and Raka[18].

In addition, Dumir and Hans-Gill[16] showed that 1 is the best constant for $n = 4$, $s = -2$; in Bambah *et al.*[3], $(7/8)^5$ is proved to be best possible for $n = 5$, $s = -1$.

In this paper we investigate these spectra for $n \geqslant 21$. We prove that, for any n, s, the best possible constants depend only on s (mod 8); for the various nonnegative spectra an upper bound of 1/5 is obtained. The statements of these results are given in Theorems 1 to 3 below.

For convenience, we set

$$E_1 = x_1^2; \; E_2 = x_1^2 + x_1 x_2 + x_2^2;$$

$$E_3 = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3;$$

$$E_5 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + (x_1 + x_2 + x_3) x_4 + (x_1 - x_2) x_5;$$

$$E_4 = E_5 (x_1, x_2, x_3, x_4, 0);$$

$$E_8 = \tfrac{1}{4} \sum_{i=1}^{4} (x_i^2 + (2x_{i+4} + x_i - \sum_{j=1}^{4} x_j)^2);$$

$$E_6 = E_8 (x_1, \ldots x_5, x_6, x_6, x_6);$$

$$\hat{E}_6 = E_8 (0, 0, x_1, \ldots, x_6).$$

These forms are our building blocks; each E_l is a positive definite integral quadratic form with minimum nonzero value equal to 1. Also,

TABLE I

f	E_1	E_2	E_3	E_4	E_5	E_6	$\hat{E}_6$	E_8
det f	2	3	4	4	4	3	4	1
$d(f)$	1	-3	-2	4	2	-3	-4	1

Define each of F_0 and H_0 to be the form which is identically zero, and for any nonzero integer m, define

$$F_m = \sum_{0 \leqslant i \leqslant m} E_8\,(x_{8i+1},\ldots,x_{8i+8}):$$

for any positive integer t, let

$$H_t = \sum_{1 \leqslant i \leqslant t} x_{2i-1}\,x_{2i}.$$

Then

$$s\,(F_m) = 8m,\ \det F_m = 1;\ s\,(H_t) = 0,\ \det H_t = (-1)^t.$$

Definition—f is said to be (integrally) equivalent to g [denoted by $f \sim g$] if there exists an integral matrix N, with det $N = \pm 1$, such that $f\,(N\,X) = g\,(X)$.

Theorem 1—Let f be an indefinite quadratic form of rank $n \geqslant 21$ and signature $s = 8q + k$, $-3 \leqslant k \leqslant 4$. Then for all $V \in R^n$

$$P^{\top}(f, V)^n / \mid D \mid \leqslant 1/\min\{\mid k \mid + 1, 4\}.$$

Moreover, the equality sign is required only when either

(i) f is equivalent to a positive multiple of sgn $(q) \cdot F_m + H_t + \tilde{g}$, for $2t = n - 8m - n(\tilde{g})$ and m and $\tilde{g}$ are given in Table II; under this equivalence V becomes O (mod 1)

TABLE II

k	0	± 1	± 2	± 2	± 3	± 3	4
s	all	all	$\mid s \mid \leqslant n-2$	$\mid s \mid = n-2$	$\mid s \mid \leqslant n-4$	$\mid s \mid > n-4$	all
m	$\mid q \mid$	$\mid q \mid$	$\mid q \mid$	$\mid q \mid -1$	$\mid q \mid$	$\mid q \mid -1$	$\mid q \mid$
$\tilde{g}$	0	$\pm E_1$	$\pm E_2$	$\mp E_6$	$\pm E_3$	$\mp E_5$	E_4

or

(ii) f is equivalent to a positive multiple of $-x_1^2 - \ldots - x_{t_1}^2 + x_{t_1+1}^2 + \ldots + x_{t_1+t_2}^2 + h$, for suitable choice of t_1 and t_2, and V becomes $(1/2, V_2)$ (mod 1), where h and V_2 are given below (Table III) :

TABLE III

k	0	± 1		± 2		± 3			4		
h	0	$\pm 2E_1$	$\mp 2E_1$	$\pm 2E_2$	$\mp 2E_6$	$\mp 4E_1$	$\pm 2E_3$	$\mp 2E_5$	$4H_1$	$2E_4$	$2E_6$
$\mathbf{V}_2$ (mod 1)	...	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	$\left(\frac{1}{2}, \frac{1}{2}\right)$	0	$\frac{1}{2}(0,1,1,1)$

This extends the following theorem proved by Watson[29]. [Note : The fourth exceptional form was not recognized until Watson[33]].

Let f be an integral, primitive, nonsingular quadratic form of rank n and signature $s(f) = 8m + k$, for $-3 \leqslant k \leqslant 4$. If f is definite, we assume it is positive definite and of rank $n \leqslant 8$. Then, if f is equivalent to none of the forms $4x_1 x_2 - x_3^2$, $16x_1 x_2 - x_3^2$, $3x_1 x_2 - x_3^2 - 3x_3 x_4 - 3x_4^2$, $x_2(6x_1 - x_2 + 3x_3) - 3x_3^2$,

$$\frac{P^+(f, 0)\, n}{|D|} \leqslant \frac{1}{\min\{|k| + 1, 4\}}.$$

The sign of equality is necessary only if either

$$|\det f| = \min\{|k| + 1, 4\} \text{ or } f \sim 8x_1 x_2 - x_3^2.$$

In Watson[33], the author finds other elements of the spectrum $\{P^+(f, 0)^n / |D| : n(f) = n\}$, for $n = 3$. In that article, he also gives an inductive argument for estimating $P^+(f, 0)^n / |D|$ for zero forms of rank $n \geqslant 4$ from information on forms of rank $n - 2$.

Theorem 2—Let f be an indefinite quadratic form of rank $n \geqslant 21$ and signature $s = 8q + k$, $-3 \leqslant k \leqslant 4$. If f is none of the exceptional forms listed in Theorem 1, then $P^+(f, \mathbf{V})^n / |D| \leqslant 1/4$. Moreover, for $|k| \leqslant 2$ equality is required only if either

(i) f is equivalent to a positive multiple of $\operatorname{sgn}(q) \cdot F_m + H_t + \tilde{g}$, for $2t = n - 8m - n(\tilde{g})$ and m and $\tilde{g}$ are given in Table IV.

TABLE IV

k	0	± 1	± 2
m	$\lvert q \rvert$	$\lvert q \rvert$	$\lvert q \rvert - 1$
$\tilde{g}$	$E_1 - E_1$; H_1	$\pm 2E_1$	$\pm(E_1 + E_1)$; $\mp \hat{E}_6$, for $s = n-2$

and under this equivalence $\mathbf{V}$ becomes $\mathbf{O}$ (mod 1) except for $\tilde{g} = E_1 - E_1$ we also have $\mathbf{V} \equiv (0, 1, 1)/2$ (mod 1);
or

(ii) f is equivalent to a positive multiple of

$$-x_1^2 - \cdots - x_{t_1}^2 + x_{t_1+1}^2 + \cdots + x_{t_1+t_2}^2 + h$$

for suitable choice of t_1, t_2, and under this equivalence V becomes $(1/2, \widetilde{V})$ (mod 1) for h and $\widetilde{V}$ given in Table V.

TABLE V

k	h	$\widetilde{V}$ (mod 1)
0	$\pm 2(E_1 + E_1)$	$1/2\,(u_1, u_2)$; $u_1 \not\equiv u_2 \pmod 2$
	$\pm 2(E_1 - E_1)$	$1/2\,(u_1, u_2)$; $u_1 \equiv u_2 \pmod 2$
	$\pm 4E_1$	$\pm 1/4$
	$\pm 2E_3$	$\pm 1/4\,(1, 1, 1)$
	$4H_1$	$1/2\,(u_1, u_2)$; $u_1 u_2 \equiv O \pmod 2$
	$2E_4$	$1/2\,(u+v, u, v, 0)$; $(u, v) \not\equiv (0, 0)$
	$\pm 2E_5$	$\pm 1/4\,(1, 1, 2, 0, 2)$
± 1	$\pm 4E_1$	0
	$\mp 2E_3$	$1/2\,(1, 1, 1)$
	$\pm 2E_5$	$1/2\,(1, 1, 0, 0, 0)$
± 2	$\mp 2(E_1 + E_1)$	$(0, 0)$
	$\mp 2(E_1 + E_1)$	$1/2\,(1, 1)$
	$\pm 2(E_1 - E_1)$	$1/2\,(0, 1)$
	$\mp 2\hat{E}_6$	0
	$\pm 2\hat{E}_6$	$1/2\,(0, 0, 1, 1, 0, 0)$

Moreover, if $P_I^+(f)^n / |D| < 1/4$, then $P_I^+(f)^n / |D| \leqslant 1/5$.

For the nonnegative spectra we have

Theorem 3—If f is an indefinite quadratic form of rank $n \geqslant 21$, then $P_I(f)^n / |D| \leqslant 1/5$.

Because of the recent work of Margulis, the restriction $n \geqslant 21$ may be a technical one, which is used only to obtain the bounds in the proof of Lemma 6. In particular, the work of Bambah *et al.*, shows that the constant given in Theorem 1 also holds for $n \geqslant 2$, $s = 0, 1, 2$; for $n \geqslant 7$, $s = -1$.

2. A Conjecture of Oppenheim (Also Called Davenport's Conjecture)

In Watson[32] the author defines the function $G(f, V)$, for fixed $V \in R^n$, to be the infimum of γ for which the inequality

$$c < f(X + V) \leqslant c + \gamma$$

is solvable in integral $X \in R^n$, for all c. Hence, for all $V \in R^n$,

$$P(f, V) \leqslant P^+(f, V) \leqslant G(f, V).$$

Setting

$$G(f) = \sup\{G(f, V) : V \in R^n\}.$$

Theorem 2 in Watson[32] shows that, for $n \geqslant 21$, $G(f)^n / |D| \leqslant 1$. Jackson[20] proves the same inequality for zero forms of arbitrary rank. More recently, Bambah *et al.*[3], proved that, for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 2 |D|^{1/n}$, the strict inequality $-\alpha < f(\mathrm{X} + \mathrm{V}) < \beta$ holds for any nonzero form f with $|s(f)| \leqslant 2$.

Restricting our attention to indefinite forms, we define

$$M(f) = \inf \{ |f(\mathrm{X})| \neq 0 : \mathrm{X} \text{ is an integral point}\}.$$

In Oppenheim[22] it was conjectured that, if $M(f) \neq 0$ and $n \geqslant 5$, then f is a rational form. In Theorem 1 of Margulies[21] it is proved that if f is an indefinite quadratic form of rank $n \geqslant 3$ which is not a multiple of an integral form, $M(f) = 0$. We shall see below that this result allows us to consider only rational forms.

Lemma 1—Let f be an indefinite quadratic form of rank $n \geqslant 3$. If f is not a multiple of a rational form, then $G(f) = 0$. Moreover, if f is rational form of rank $n \geqslant 5$ and V is not a rational point, then $G(f, \mathrm{V}) = 0$.

PROOF: We first suppose that f is not a rational form. Then, by Oppenheim's conjecture, $M(f) = 0$. In Oppenheim[23] it is shown that, for $n \geqslant 3$, $P^+(f, \mathbf{0}) = 0$ implies $P^+(-f, \mathbf{0}) = 0$; hence, from $M(f) = 0$ we obtain $P^+(f, \mathbf{0}) = 0$. In Theorem 1 of Watson[30] it is proved that $G(f) = 0$ follows from $P^+(f, \mathbf{0}) = 0$. This proves the first assertion.

If f is an indefinite integral quadratic form with $n \geqslant 5$, then by Meyer's Theorem [Cassels[7], p. 75, Corollary 1], f represents zero nontrivially. Hence, by Theorem 2 in Watson[30], $G(f, \mathbf{V}) = 0$ for any nonrational point $\mathbf{V}$.

3. Results on Equivalence

In this section f and g denote nonsingular primitive integral (not necessarily indefinite) quadratic forms of respective ranks $n(f)$, $n(g)$, and respective signatures $s(f)$, $s(g)$. Because in any of our spectra the values for two equivalent forms are equal, it suffices to consider a complete set of representatives for the equivalence classes. It has been proved, for instance Theorem 1.1 in Cassels[7], that the number of equivalence classes of forms of fixed rank and discriminant is finite. Because results of Watson[32] will allow us, for any fixed n, to consider only a finite set of discriminants, our first step will be to determine a set of representatives for forms with the discriminants which occur in our problem. The following notions are useful in this determination.

Definition—For prime p, f is said to be p-adically equivalent to g (denoted by $f \tilde{p} g$) if, for each positive integer t, there exists an integral matrix N such that p does not divide det N and $f(N\mathrm{X}) \equiv g(\mathrm{X}) \pmod{p^t}$.

Definition—Let $d(f) = d(g) = d$. Then f is said to be congruentially equivalent to g (denoted by $f \cong g$) if $f \tilde{p} g$, for all primes p dividing d.

Note : In Watson[31], Theorem 43 it is shown that if $f \cong g$ then $s(f) \equiv s(g)$ (mod 8).

Lemma 2—Let f and g be indefinite forms of the same rank $n \geqslant 3$, and the same discriminant d. If, for any integer $m \geqslant 5$, d is not divisible by $m^{n(n-1)/2}$, then $f \sim g$ if and only if $f \cong g$ and $s(f) = s(g)$.

PROOF : Watson[31], Corollary 2, p. 111.

We shall later prove that all of our discriminants are small, relative to the hypothesis of Lemma 2.

Lemma 3—Let p be any prime dividing the integer d. Then there are only finitely many p-adic equivalence classes of fixed rank and discriminant d. Each class is represented by a disjoint sum

$$f_0 + pf_1 + \ldots + p^k f_k \qquad \ldots(4)$$

where, for each $0 \leqslant i \leqslant k$, either $f_i = 0$ or there exists $t_i \geqslant 0$ such that $f_i = H_{t_i} + g$, where either $g = 0$ or, if $p = 2$,

$$g = E_2,\ \pm cx^2,\ x^2 \pm cy^2,\ -x^2 - cy^2 \text{ for } c = 1, -3;$$

if p is odd,

$$g = x^2,\ x^2 - by^2,\ bx^2$$

where b is quadratic nonresidue (mod p).

PROOF : Watson[31], Theorems 32 and 35, p. 54 ff.

We note that since the sum in (4) is disjoint, the determinant of the form in (4) is $\det f_0 \,.\, \det(pf_1) \ldots\ldots \det(p^k f_k)$.

Lemma 4—Let $n(f) = n(g)$ and p be any prime. Then $f \tilde{p} g$ if and only if $\det f \cdot \det g$ is a p-adic square, $p^t \| d(f), d(g)$, and there exists an integral matrix N such that p does not divide $\det N$ and $f(N\mathbf{X}) \equiv g(\mathbf{X}) \pmod{p^t}$.

PROOF : We note that the quotient $\det f \cdot \det g / (-1)^{2[n/2]} d(f) \cdot d(g)$ equals 1 or 4. This lemma thus follows from the analogues for determinants given in Watson[31] the sufficiency is given in Theorem 33 (ii); the necessity can be found on p. 50.

The last result will be used to eliminate replications in the list obtained from Lemma 3. To obtain the congruential classes for a fixed rank and discriminant d, the information from the p-adic classes for all p dividing d must be combined.

4. A REDUCTION OF OUR PROBLEM

We recall that in Lemma 1 we showed if the rank of f is at least 5, and either f is not a multiple of a rational form or $\mathbf{V}$ is not a rational point, then $G(f, \mathbf{V}) = 0$.

We now return to the function $G(f, \mathbf{V})$ and restrict our attention to primitive integral forms f and rational points $\mathbf{V}$.

Writing $\mathbf{V} = q^{-1}(u_1, \ldots, u_n)$, where $gcd(u_1, \ldots, u_n) = 1$, expansion of $f(\mathbf{X} + \mathbf{V})$ yields $f(\mathbf{X} + \mathbf{V}) = f(\mathbf{X}) + 1/c_1\, L(\mathbf{X}) + r$, where r is a rational number and c_1 is the integer such that L is a linear integral form in which any common divisor of its coefficients is relatively prime to c_1.

Lemma 5—q divides $c_1\, D$.

PROOF : For $f(\mathbf{X}) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij}\, x_i\, x_j$, expansion of $f(\mathbf{X} + \mathbf{V})$ yields the linear form

$$L = \frac{c_1}{q} \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij}\, u_i\, x_j$$

where

$$b_{ij} = \begin{cases} 2\, a_{ij} & \text{for } i = j \\ a_{ij} & \text{for } i \neq j. \end{cases}$$

Setting $N_j = \frac{c_1}{q} \sum_{i=1}^{n} b_{ij}\, u_i$, the fact that $L(X)$ is an integral form implies that N_j is an integer, for all $1 \leqslant j \leqslant n$. Thus, for $\mathbf{N} = (N_1, \ldots, N_n)$, $\mathbf{U} = (u_1, \ldots, u_n)$, and $B = (b_{ij})$, we have $q\, \mathbf{N} = c_1\, \mathbf{U}\, B$.

Since $\det B = \det f \neq 0$, then by Cramer's Rule

$$u_i = \frac{qB_i}{c_1 \det f}$$

where B_i is the correct minor of B augmented by $\mathbf{N}$. We recall that $\mathbf{U}$ is an integral point with $gcd(u_1, \ldots, u_n, q) = 1$ and so q divides $c_1 \det f$.

The following lemma follows from Theorem 3 of Watson[32]. Watson's proof uses the restriction $n \geqslant 21$; in light of Margulis' work a modification of Watson's proof should allow a weakening of $n \geqslant 21$. As noted in Watson[32], p. 567 the result may be true for all $n \geqslant 9$, but there is a counterexample for all $n \leqslant 8$; namely,

$$f = x_1^2 \pm 2(x_2 + 1/2)^2 \pm \cdots \pm 2(x_n + 1/2)^2$$

has $|\det A| = 2^{2n-1}$, $f(\mathbf{X}) \equiv (s \pm 1)/2 \pmod 4$, for all integral $\mathbf{X}$.

Lemma 6—Let $n \geqslant 21$. If $G(f, \mathbf{V})^n / |D| > 1/5$, then $c_1 = 1$ and one of the following holds :

(i) $G(f, \mathbf{V}) = 1$ and $|D| < 5$;

(ii) $G(f, \mathbf{V}) = 2$, $|D| < 5 \cdot 2^n$, and $f \equiv L^2 \pmod 2$.

Recalling that $G(f, V) \geqslant P^+(f, V), > P(f, V)$, our analysis of each spectrum above 1/5 will be divided into the cases given in Lemma 6.

5. Proof of the Theorems for the case $G(f, V) = 1$.

We consider primitive integral forms f with $n \geqslant 21$ and rational points $V = \frac{1}{q} U$ for which $G(f, V) = 1$.

Lemma 7—If $n \geqslant 5$ and $P^+(f, V)^n / |D| > 1/5$, then $P^+(f, V) = 1$.

Proof : Since $G(f,V) \geqslant P^+(f,V)$, then $G(f,V)^n / |D| > 1/5$ and the hypotheses of Lemma 6 hold. Thus, $c_1 = 1$ and from Lemma 5 we obtain that q divides D. Hence, the denominator of the rational number $P^+(f, V)$ divides D^2. Hence, either $P^+(f, V) = 1$ or $P^+(f, V) \leqslant 1 - 1/D^2$.

If $P^+(f, V) \neq 1$, then $P^+(f, V) \leqslant 1 - 1/D^2$. Setting $T(|D|, n) = (1 - D^{-2})^n / |D|$, we obtain $P^+(f, V)^n / |D| \leqslant T(|D|, n)$.

Since $n \geqslant 21$ and $1 \leqslant |D| \leqslant 4$, $T(|D|, n)$ is a decreasing function of n and an increasing function of $|D|$. Therefore, $T(|D|, n) \leqslant T(4, 21) < 1/5$, contrary to hypothesis.

Lemma 8—Let $t \geqslant 1$ be an integer, p be an odd prime, and b be any quadratic nonresidue (mod p). If d_1 and d_2 are integers which are not divisible by p, then

$$d_1 x_1^2 + p^t d_2 x_2^2 \underset{p}{\sim} c(kx_1^2 + p^t x_2^2)$$

where $cd_2 \equiv 1 \pmod{P^t}$ and $k = 1$ or b is chosen so that $kd_1 d_2$ is a p-adic square.

Proof : For convenience, we define the forms

$$g_1(X) = d_1 x_1^2 + p^t d_2 x_2^2 \text{ and } g_2(X) = c(kx_1^2 + p^t x_2^2)$$

for any c, k as given in the conclusion. Since $kd_1 d_2$ a p-adic square, there exists m such that $kd_1 d_2 \equiv m^2 \pmod{p^t}$. Also,

det $g_1 \cdot$ det $g_2 = 16\, c^2 kd_1 d_2 p^{2t}$ is a p-adic square and $p^t \| d(g_1), d(g_2)$. For $N = \begin{pmatrix} mk^{-1} & 0 \\ 0 & d_2 \end{pmatrix}$, where det N is not divisible by p,

$$\begin{aligned} g_2(NX) &= c(k(mk^{-1} x_1)^2 + p^t (d_2 x_2)^2) \\ &= cm^2 k^{-1} x_1^2 + p^t cd_2^2 x_2^2, \end{aligned}$$

where

$$c\, m^2 k^{-1} \equiv cd_1 d_2 \equiv d_1 \pmod{p^t}$$

and

$$cd_2^2 \equiv d_2 \pmod{p^t}.$$

Hence,

$g_2(N\,\mathbf{X}) \equiv g_1(\mathbf{X}) \pmod{p^t}$ and, by Lemma 4, $g_1 \underset{p}{\sim} g_2$.

Lemma 9 – Let f be a primitive integral quadratic form with $n(f) = n \geqslant 4$. If $|D| \leqslant 4$, then, for some g as given in Table VI, $f \cong H_m + g$, where the sum is disjoint and $2m = n - n(g)$.

TABLE VI

$\lvert D \rvert$	1	2	3	4
g	0	$\pm E_1$	$\pm E_2$	$E_1 \pm E_1$; $E_1 - E_1$; $2H_1$; E_4; $\pm 2E_1$; $\pm E_3$

Moreover, these forms are pairwise incongruent.

PROOF: If $|D| = 1$, then the integralness of d implies that n is even. Since n is even, then $d \equiv 0, 1 \pmod 4$ by (3) and so $d = 1$. Moreover, $d(H_m) = 1$ implies that $f \cong H_m$ by definition of congruential equivalence.

If $|D| = 2$ and n were even, then $|D| = |d| \equiv 2 \pmod 4$, a contradiction to $d \equiv 0, 1 \pmod 4$. Hence, n is odd and $|d| = 1$. Also, $d(H_m \pm E_1) = \pm 1$. Therefore, choosing the sign so that $d = d(H_m \pm E_1)$, we have that $f \cong H_m \pm E_1$.

If $|D| = 3$, the integralness of d again implies that n is even and so $d = -3 = d(\pm E_2) = d(H_m \pm E_2)$. Hence, it is sufficient to show that $f \underset{3}{\sim} H_m \pm E_2$. By Lemma 3, $f \underset{3}{\sim} f_0 + 3f_1$. Also, Lemma 4 implies $3 \parallel d(f_0 + 3f_1)$ and so $n(f_1) = 1$. Since n is even, $n(f_0)$ must be odd and

$$f \underset{3}{\sim} H_m + d_1 x_{n-1}^2 + 3d_2 x_n^2$$

for some $d_1, d_2 \not\equiv 0 \pmod 3$. Applying Lemma 8 we obtain

$$f \underset{3}{\sim} H_m + c(k x_{n-1}^2 + 3x_n^2)$$

for $cd_2 \equiv 1 \pmod 3$ and $k = \pm 1$ chosen so that $kd_1 d_2$ is a padic square. For the choice $k = -1$, we have

$$d(f) \cdot d(H_m + c(-x_{n-}^2 + 3x_n^2)) = -36c^2$$

which is not a 3-adic square. Therefore,

$$f \underset{2}{\sim} H_m \pm (x_{n-1}^2 + 3x_n^2).$$

For $N = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, we obtain $E_2(N\mathbf{X}) = x_1^2 + 3x_2^2$. Since $\det N = -2$, by Lemma 4 $f \underset{3}{\sim} H_m \pm E_2$, proving $f \cong H_m \pm E_2$.

For the case $|D| = 4, f \underset{2}{\sim} f_0 + 2f_1$, Lemma 4 implies that $4 \parallel \det(f_0 + 2f_1)$. Hence, $n(f_1) \leqslant 2$ and each of f_0, f_1 is of the form $H_m + g$, where

$$g = 0,\ E_2,\ \pm cE_1,\ E_1 \pm cE_1, \text{ or } -E_1 - cE_1, \text{ for } c = 1, -3.$$

Moreover, $4 \| 2^{n}(f_1) \det f_0 \cdot \det f_1$ restricts f_1 to $0, H_1, E_2, \pm cE_1$. We consider each of these cases in Table VII, where we let $R = 0, 1, 2$ be such that $2^R \| \det f_0$.

TABLE VII

f_1	R	f_0	$\det f \cdot \det(f_0 + 2f_1)$	c	$f_0 + 2f_1$
0	2	$H_m + E_1 \pm cE_1$	$\pm 16c$	1	$H_m + E_1 \pm E_1$
		$H_m - E_1 - cE_1$	$\pm 16c$	1	$H_m - E_1 - E_1$
H_1	0	H_m	± 16	—	$H_m + 2H_1$
		$H_m + E_2$	± 48	—	—
E_2	0	H_m	± 48	—	—
		$H_m + E_2$	± 144	—	$H_m + E_2 + 2E_2$
$\pm cE_1$	0	H_m	$\pm 16c$	1	$H_m \pm 2E_1$
		$H_m + E_2$	$\pm 48c$	-3	$H_m + E_2 \pm 6E_1$

We next apply Lemma 4 to show that $E_2 + 2E_2 \cong E_4$ and $E_2 \pm 6E_1 \cong E_3$. We recall that $2^t \| d(f)$, set $g = f_0 + 2f_1$, and note that

TABLE VIII

g	$d(g)$	t	N	$\det N$	$g(NX) \pmod{2^t}$
$E_2 + 2E_2$	36	2	$\begin{pmatrix} 2 & 2 & 0 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$	-1	$E_4(X)$
$E_2 \pm 6E_1$	18	1	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1	$E_3(X)$

Hence, $E_2 + 2E_2 \cong E_4$ and $E_2 \pm 6E_1 \cong E_3$ and Table VI follows from Table VII.

Finally, we note that $s(H_m + g) = s(g)$. Since the signatures of congruent forms are in the same congruence class (mod 8), to complete the proof of Lemma 9 it suffices to check that there exists no matrix N of odd determinant such that $2H_1(N\mathbf{X}) \equiv (E_1 - E_1)(\mathbf{X}) \pmod 4$.

Lemma 10—Let f be a primitive integral quadratic form with $n(f) \equiv n \geqslant 4$ and $s(f) \equiv k \pmod 8$, where $-3 \leqslant k \leqslant 4$. If $|D| \leqslant 4$, then $f \cong H_m + g$, where $2m = n - n(g)$, $s(g) = k$, and g is given in Table IX

TABLE IX

k	0	± 1	± 2	± 3	4
g	$0; E_1 - E_1; 2H_1$	$\pm E_1; \pm 2E_1$	$\pm E_2; \pm (E_1 + E_1)$	$\pm E_3$	E_4

Moreover, the forms are incongruent.

PROOF : From Lemma 9 we obtain $f \cong H_m + g$, for $2m = n - n(g)$ and g listed in Table VI. Since $s(H_m) = 0$ and $f \cong H_m + g$, then $s(f) \equiv s(H_m + g) = s(g)$ (mod 8). For all g listed in Table VI we have $-3 \leqslant k \leqslant 4$. Hence, $s(g) = k$ and Lemma 10 is proved.

Proposition 1—Let f be a primitive integral quadratic form with $n(f) = n \geqslant 5$, $|D| \leqslant 4$, and $s(f) = 8q + k$, where $-3 \leqslant k \leqslant 4$. As in Lemma 10, $f \cong H_m + g$, where $2m = n - n(g)$ is given in Table IX. Let $2t = n - n(g) - 8|q|$.

(i) If $t \geqslant 0$, then $f \sim \operatorname{sgn}(q) \cdot F_{|q|} + H_t + g$, where g is given in Table IX.

(ii) If $t < 0$, then $f \sim \operatorname{sgn}(q) \cdot F_{|q|-1} + h$, where h is given in Table X.

TABLE X

k	t	h
± 2	-1	$H_1 \mp E_6$; $H_1 \mp \hat{E}_6$
± 3	$-1; -2$	$H_{t+3} \mp E_5$
4	$-1; -2; -3$	$H_{t+} - E_4$

Moreover, these forms are pairwise inequivalent.

PROOF : Since equivalent forms are congruent and have the same signature, the given forms are pairwise inequivalent.

Applying Lemma 10 we have that $f \cong H_{4|q|+t} + g$, where g is given in Table IX. We note that since $d(\pm E_8) = d(\pm H_4) = 1$ and $n(\pm E_8) = n(\pm H_4) = 8$, by definition we have $\pm E_8 \equiv \pm H_4$ and so for all $r \geqslant 0$, $\pm F_r \equiv \pm H_{4r}$.

Considering the case when $t \geqslant 0$, we have $f \cong \operatorname{sgn}(q). F_{|q|} + H_t + g$ with $s(\operatorname{sgn}(q) \cdot F_{|q|} + H_t + g) = 8q + s(g) = 8q + k = s(f)$. From Lemma 2 we thus have $f \sim \operatorname{sgn}(q) \cdot F_{|q|} + H_t + g$.

We next note that $|s(f)| = 8|q| \mp |s(g)|$. If $t < 0$, then $q \neq 0$ and

$$-2 \geqslant 2t = n - n(g) - 8|q| = n - n(g) - (|s(f)| \pm |s(g)|)$$
$$= n - s(f) - (n(g) \pm |s(g)|) \geqslant 2 - (n(g) \pm |s(g)|)$$

Since f is indefinite. In particular, $n(g) \pm |s(g)| \geqslant 4$. Considering the choices of g, we thus have that $|s(g)| \geqslant 2$ and $|s(f)| = 8|q| - |s(g)|$. An inspection of the forms in Table IX, combined with $2t = n(f) - n(g) - (|s(f)| + |s(g)|)$, yields

TABLE XI

k	g	$n(f) - \mid s(f) \mid \leqslant$	$d(f)$	t
± 2	$\pm E_2; \pm (E_1 + E_1)$	2	$-3; -4$	-1
± 3	$\pm E_3$	4	-2	$-1; -2$
4	E_4	6	4	$-1; -2; -3$

Hence, when $t < 0, f \cong H_{4|q|+t} + g \cong \operatorname{sgn}(q) \cdot F_{|q|-1} + H_{t+4} + g$, for g given in Table XI.

To prove that $f \sim \operatorname{sgn}(q). F_{|q|-1} + h$, for the forms h given in Table X it thus suffices to show that

$$H_{t+4} + g \cong h \text{ and } s(h) = k - 8 \cdot \operatorname{sgn}(k) \qquad \ldots(5)$$

the latter since we want $s(f) = 8 \cdot \operatorname{sgn}(q) \cdot (\mid q \mid - 1) + s(h)$; $s(h) = (8q + k) - 8q + 8 \cdot \operatorname{sgn}(q) = k + 8. \operatorname{sgn}(q)$. We recall that $\mid 8q + k \mid = \mid s(f) \mid = 8 \cdot \mid q \mid - \mid k \mid$ and so $q < 0$ if and only if $k > 0$. Hence, $k + 8 \cdot \operatorname{sgn}(q) = k - 8 \cdot \operatorname{sgn}(k)$, which gives the second condition of (5).

We now consider the positive definite forms $E_6, \hat{E}_6, E_5, E_4$ with respective discriminants $-3, -4, 2, 4$. By Lemma 10, each of these is congruent to $H_m, + g$, for appropriate $m \geqslant 0$ and some g in Table IX. An analysis of discriminants, ranks, and signature (mod 8) yields

$$E_6 \cong H_2 - E_2; \hat{E}_6 \cong H_2 - (E_1 + E_1); E_5 \cong H_1 - E_3; E_4 \equiv \pm E_4.$$

The conclussons of the proposition thus follow from our above analysis and the information in Table XI.

Proposition 2—Let f be a primitive integral quadratic form of rank $n \geqslant 21$ and let $\mathbf{V} \in R^n$ for which $G(f, \mathbf{V})^n / \mid D \mid > 1/5$. If $G(f, \mathbf{V}) = 1$, then $f \sim \pm F_r + H_v + \tilde{g}$, for the appropriate choice of sign, and $r, v, \tilde{g}$ given in Proposition 1. Under this equivalence $\mathbf{V}$ becomes $(\mathbf{0}, \tilde{\mathbf{V}})$ for $\tilde{\mathbf{V}} \in R^{n(g)}$. Moreover, $\{f(\mathbf{X} + \mathbf{V}) : \mathbf{X}$ is an integral point$\} = \tilde{g}(\tilde{\mathbf{V}}) + Z$.

PROOF : Since $G(f, \mathbf{V})^n / \mid D \mid > 1/5$ and $G(f, \mathbf{V}) = 1$, by Lemma 6 we have $\mid D \mid \leqslant 4$ and Proposition 1 implies that $f \sim \pm F_r + H_v + \tilde{g}$, for the appropriate choice of sign, and $r, v, \tilde{g}$. We write $\mathbf{V} = q^{-1}(u_1, \ldots, u_n)$, where $gcd(u_1, \ldots, u_n) = 1$, and as in Lemma 5 $f(\mathbf{X} + \mathbf{V}) = f(\mathbf{X}) + L(\mathbf{X})/c_1 + r$, where L is an integral linear form and, by Lemma 6, $c_1 = 1$. Since $\pm F_r + H_v + \tilde{g}$ is a disjoint sum and the statement of this proposition only gives information about $\pm F_r + H_v$, it is likewise sufficient to consider only E_8 and H_1. But

$$H_1(x + u/q, y + u'/q) = H_1(x, y) + q^{-1}(u'x + uy) + uu'/q^2$$

and the linear portion is an integral form; for all $8r \leqslant i \leqslant 2m$, $v_i \equiv O \pmod 1$. Expanding $E_8(\mathbf{X} + \mathbf{V})$ we obtain that the linear portion equals

$$\frac{1}{q}\left[\sum_{i=1}^{4}\left(2u_i + \sum_{j\neq i}(u_j - u_{j+4})\right)x_i + \sum_{i=1}^{4}\left(2u_{i+4} - \sum_{j\neq i} u_j\right)x_{i+4}.\right]$$

Since this is an integral form,

$$q \text{ divides } 2u_i + \sum_{j\neq i}(u_j - u_{j+4}) \qquad \ldots(6)$$

and

$$q \text{ divides } 2u_{i+4} - \sum_{j\neq i} u_j \qquad \ldots(7)$$

for all i. Summing over i in (7) we obtain

$$q \text{ divides } 2U_1 - 3U_2 \qquad \ldots(8)$$

where

$$U_1 = \sum_{i=1}^{4} u_{i+4} \text{ and } U_2 = \sum_{i=1}^{4} u_i.$$

We recall from Lemma 5 that q divides $c_1 D = D$ and $|D| \leqslant 4$. Therefore, 2 and 3 are the only prime divisors of q.

If 3 divides q, then (8) implies that 3 divides U_1. Adding (6) and (7)

$$q \text{ divides } 2u_i - U_1 + 3u_{i+4}$$

and so 3 divides each u_i. Hence, from (6) we obtain that 3 divides each $U_1 - u_{i+4}$. Therefore, if 3 divides q, then

$$u_i \equiv O \pmod 3 \text{ for all } i \leqslant |8r|.$$

If q is divisible by a power of 2, then (8) implies that U_2 is even. Subtracting (7) from (6) we thus have that $U_1 + u_{i+4}$ is even;

$$\sum_{i=1}^{4}(U_1 + u_{i+4}) = 5U_1 \text{ is even;}$$

and so u_{i+4} is even, for all $1 \leqslant i \leqslant 4$ The evenness of u_i, for $i \leqslant 4$, follows from (6) and the evenness of each u_{j+4} and U_2. Hence, there exist integers w_i such that $u_i = 2w_i$, for all $i \leqslant 8$. If 4 divides q, then (6), (7), (8) can be rewritten in terms of the w_i; the above argument applies and yields $w_i \equiv O \pmod 2$, for all $i \leqslant 8$.

Therefore, $u_i \equiv O \pmod q$ for all $i \leqslant 2m$ and we write $\mathbf{V} = (0, \tilde{\mathbf{V}})$, where $\tilde{\mathbf{V}} \in R^{n(g)}$.

To complete the proof, we show that

$$\{f(\mathbf{X}+\mathbf{V}) : \mathbf{X} \text{ is integral}\} = \tilde{g}(\tilde{\mathbf{V}}) + \mathbb{Z}. \qquad \ldots(9)$$

Writing $\mathbf{X} = (\mathbf{Y}, \mathbf{Z})$, where $\mathbf{Z} \in \mathbb{Z}^{n(\tilde{g})}$, we have $\mathbf{X} + \mathbf{V} = (\mathbf{Y}, \mathbf{Z} + \tilde{\mathbf{V}})$. We recall, from Proposition 1, that $n(\tilde{g}) \leqslant 6$ and so $|8r| + 2v \geqslant 15$, implying that either $|r| > 0$ or $v > 0$.

If $t \neq 0$ then $\{(\pm F_r + H_v)(\mathbf{Y}) : \text{integral } \mathbf{Y}\} = \mathbb{Z}$ and, for fixed $\mathbf{Z}$, $\{f(\mathbf{Y}, \mathbf{Z} + \tilde{\mathbf{V}}) : \mathbf{Y} \text{ is integral}\} = \tilde{g}(\mathbf{Z} + \tilde{\mathbf{V}}) + \mathbb{Z}$. $G(f, \mathbf{V}) = 1$ thus implies (9).

On the other hand, if $v = 0$ and $s(\pm F_r) > 0$, then $\{\pm F_r(\mathbf{X}) : \text{integral } \mathbf{X}\} = \mathbb{N}^*$, the set of nonegative integers.

Therefore, for fixed $\mathbf{Z} \in \mathbb{Z}^{n(\tilde{g})}$, $\{f(\mathbf{Y}, \mathbf{Z} + \tilde{\mathbf{V}}) : \text{integral } \mathbf{Y}\} = g(\mathbf{Z} + \tilde{\mathbf{V}}) + \mathbb{N}^*$. From $G(F, \mathbf{V}) = 1$ we thus obtain

$$\tilde{g}(\mathbf{Z} + \tilde{\mathbf{V}}) - \tilde{g}(\tilde{\mathbf{V}}) \in \mathbb{Z}, \text{ for all } \mathbf{Z} \in \mathbb{Z}^{n(g)}.$$

Moreover, since f is indefinite and $\{F_r(\mathbf{Y}) : \text{integral } \mathbf{Y}\} = \mathbb{N}^*$, there exist integral vectors $\mathbf{Z}_i$ such that $\tilde{g}(\mathbf{Z}_i + \mathbf{V}) \xrightarrow[i \to \infty]{} -\infty$. Writing $\tilde{g}(\mathbf{Z}_i + \mathbf{V}) - \tilde{g}\,\mathbf{V})$ $= n_i$, we have, for fixed i, $\{f(\mathbf{Y}, \mathbf{Z}_i + \mathbf{V}) : \text{integral } \mathbf{Y}\} = \tilde{g}(\tilde{\mathbf{V}}) + n_i + \mathbb{N}^*$, from which (9) follows. An analogus argument applies for $s(\pm F_r) < 0$.

Proof of the Theorems of the case $G(f, V) = 1$

We assume that $P^+(f, \mathbf{V})^n / |D| > 1/5$ and $G(f, \mathbf{V}) = 1$. By Proposition 1, $f \sim F_r + H_v + \tilde{g}$ for some r, v and $\tilde{g}$ as given in Proposition 1. Moreover, from Proposition 2. under this equivalence we have $\mathbf{V} = (\mathbf{0}\ \tilde{\mathbf{V}})$. We claim that all non-integral $\mathbf{V}$ which satisfy $G(f, \mathbf{V}) = 1$ are given in Table XII.

Column 4 (Table XII) follows from Lemmas 5 and 6. We shall complete the argument for $\tilde{g} = \pm E_3$. Recalling (9), we obtain that, for any integral $\mathbf{N} = (n_1, n_2, n_3)$, $E_3(\tilde{\mathbf{V}} + \mathbf{N}) - E_3(\tilde{\mathbf{V}})$ is an integer. Setting $\tilde{\mathbf{V}} = 1/4\,(u_1, u_2, u_3)$ we thus obtain that

$$u_1(2n_1 + n_2 + n_3) + u_2(n_1 + 2n_2 + n_3) + u_3(n_1 + n_2 + 2n_3) \equiv O \pmod 4$$

for all choices of integral $\mathbf{N}$. Using $\mathbf{N} = (1, 0, 0), (0, 1, 0), (0, 0, 1)$ we thus obtain that $u_1 \equiv u_2 \equiv u_3 \equiv 1 \pmod 4$, completing the argument for $\tilde{g} = \pm E_3$.

Continuing with the proof of the theorems for the case when $G(f, \mathbf{V}) = 1$, we assume that $P^+(f, \mathbf{V})^n / D| > 1/5$, where

TABLE XII

k	$\pm\tilde{g}$	$\lvert D \rvert$	$q\neq 1$	$\tilde{V}$ (mod 1)
0	0	1	—	—
	$E_1 - E_1$; $2H_1$	4	2, 4	$\frac{1}{2}(u_1, u_2)$
± 1	E_1	2	2	$\frac{u}{2}$
	$2E_1$	4	2, 4	[illegible]/4
± 2	E_2	3	3	[illegible]/3 (1, 1)
	$E_1 + E_1$	4	2, 4	$\frac{1}{2}(u_1, u_2)$
	E_6	3	3	$\frac{u}{3}(0, 1, 1, 1, 0, -1)$
	$\hat{E}_6$	4	2, 4	$\frac{1}{2}(0, 0, u, v, u+v, u+v)$
± 3	E_3	4	2, 4	[illegible]/4 (1, 1, 1)
	E_5	4	2,4	$\frac{u}{4}(1, 1, 2, 0, 2)$
4	E_4	4	2, 4	$\frac{1}{2}(u+v, u, v, 0)$

TABLE XIII

k	$\pm\tilde{g}$	$\pm\tilde{g}\,(\tilde{V})$	$\tilde{V}$ (mod 1)	$P^+(f, V)^n / \lvert D \rvert$
0	0	—	0	1
	$E_1 - E_1$	$\frac{1}{4}(u_1^{2} - u^{2})$	$\frac{u}{2}(1, 1)$	1/4
	H_1	$\frac{1}{4}(u_1 u_2)$	0	1/4
± 1	E_1	$u^2/4$	0	1/2
	$2E_1$	$u^2/8$	0	1/4
± 2	E_2	$u^2/3$	0	1/3
	$E_1 + E_1$	$\frac{1}{4}(u_1^2 + u_2^2)$	0	1/4
	E_6	$2u^2/3$	0	1/3
	$\hat{E}_6$	$\frac{3}{4}(u^2+v^2)$	0	1/4
± 3	E_3	$3u^2/8$	0	1/4
	E_5	$5u^2/8$	0	1/4
4	E_4	$\frac{1}{2}(u^2+v^2+uv)$	0	1/4

$$P^+(f, V) = \inf\{fX + V) > 0 : \text{integral } X\}.$$

By Lemma 7, $P^+(f, V)^n / |D| > 1/5$ implies that

$P^+(f, V) = 1; \tilde{g}(\tilde{V}) \in \mathbb{Z}$, by (9). Using this and Table XII, we obtain column 5 Table XIII below and so Theorems 1 and 2 for the case when $G(f, V) = 1$.

We note that $P(f, V)^n | D | > 1/5$ implies that $P^+(f, V)^n / | D | > 1/5$. Therefore, if $G(f, V) = 1$ and $P(f, V)^n / | D | \geqslant 1/5$, then the above argument applies with $\tilde{V}$ and $\tilde{g}$ as listed in Table XI. Since

$$P(f, 0) = \inf\{f(X) \geqslant 0 : X \text{ is nonzero integral point}\} = 0$$

for any of these f, then $P_l(f)^n | D | \leqslant 1/5$ for all f and Theorem 3 is proved for the case when $G(f, V) = 1$.

6. Proof of the Theorems for the case $G(f, V) = 2$

By Lemma 6, we have that $f \equiv L^2 \pmod 2$ and so, for $i \neq j$, a_{ij} is even. By definition, the diagonal entries of the matrix A are even.

Lemma 11—If $f \underset{2}{\sim} g = f_0 + 2f_1$, then $f_0 \cong cx_1^2$ or $\pm x_1^2 + cx_2^2$.

Proof Since $f \underset{2}{\sim} g$, by definition there exists an integral matrix M with odd determinant such that $g(MX) \equiv f(X) \pmod 2$. A comparison of coefficients yields that, for $y_l = m_{ij}x_1 + \ldots + m_{in}x_n$, $f_0(Y) \equiv f_0(X) \pmod 2$. If $f_0 \cong E_2$ or there exists $t > 0$ such that $f_0 \cong H_t + g_0$, then for $i \neq j$

$$m_{1l}m_{2j} + m_{1j}m_{2l} \equiv a_{lj} \pmod 2.$$

Since a_{lj} is even, then also

$$m_{1l}m_{2j} - m_{1j}m_{2t} \equiv O \pmod 2.$$

Letting M_{lj} be the $n-2$ order minor of M with the first two rows and the i, j columns removed, then (cf. Hadley[17], page 98, eqn. (3-100))

$$\det M = \sum_{i \neq j} \epsilon_{ij}(m_{1l}m_{2j} - m_{1j}m_{2l}) \det M_{lj}.$$

for the correct choice of $\epsilon_{lj} = \pm 1$, contrary to the oddness of det M.

Lemma 12—Let $n \geqslant 5$ and let a_{ij} be even, for all $i \neq j$. If $| D | \leqslant 2^{n+2}$, then $f \sim \pm x_1^2 + g$, where the sign can be chosen so that g is an indefinite form of rank $n - 1$.

Proof : We first show that the result follows if f represents both ± 1. Let **B** be an integral point for which $f(\mathbf{B}) = \pm 1$. Then **B** is primitive and **B** can be extended to a basis, thus obtaining a unimodular matrix B in which **B** is the first column, Thus,

$$f(B\,\mathbf{X}) = \pm\; x_1^2 + x_1\, M(x_2, \ldots, x_n) + h(x_2, \ldots, x_n),$$

where h is a quadratic form and M is the linear form given by

$$M(\mathbf{X}) = \sum_{i=1}^{n} 2\, a_{ii}\, b_{i1} \sum_{k=2}^{n} b_{ik}\, x_k + \sum_{i<j}^{n} a_{ij} \sum_{k=2}^{n} (b_{i1}\, b_{jk} + b_{ik}\, b_{j1})\; x_k$$

Since a_{ij} is even for all $i \neq j$, then M has even coefficients, say

$$M = 2\,(c_2\, x_2 + \ldots + c_n\, x_n).$$

Hence,

$$f(B\,\mathbf{X}) = \pm\,(x_1 \pm (c_2\, x_2 + \ldots + c_n\, x_n))^2 + (h \mp M^2/4)\,(x_2, \ldots, x_n):$$

$f \sim \pm\; x_1^2 + g$, for the quadratic form $g = h \mp M^2/4$. The sign of $\pm\,1$ is chosen so that g is indefinite.

In order to complete the proof of the lemma it suffices to show that f represents both $\pm\,1$. For this we shall use Theorem 51 (ii) in Watson[33] : Let a be a fixed integer. If there exists a primitive solution $\mathbf{X}$ to $f(\mathbf{X}) \equiv a \pmod{d}$, then there exists a form h which is congruentially equivalent to f, has the same signature, and properly represents a.

We note that $|\,D\,| \leqslant 2^{n+2}$ and Lemma 2 imply that such h is equivalent to f and hence that f itself properly represents a. Moreover, from $f \equiv L^2 \pmod{2}$ we obtain that 2^n divides D and so $|\,D\,| = r.\,2^n$, for some $r \leqslant 4$.

For the case when $r = 3$, by Lemma 3 we write $f \underset{3}{\sim} g_0 + 3g_1$. Lemma 4 implies that $n(g_1) = 1$ and so $n(g_0) \geqslant 4$. Inspecting the choices in Lemma 3, we observe that $g_0 \cong H_t + g$, for some $t \geqslant 1$. Therefore, g_0 represents $\pm\,1$; that is, each of $f(\mathbf{X}) \equiv \pm\,1 \pmod{3}$ is solvable. Suppose we have shown that, for each of $\epsilon = \pm\,1$, there exists primitive $\mathbf{P}$ such that $f(\mathbf{P}) = \epsilon + 2^n\, m$. For primitive $\mathbf{Q}$ such that

$f(\mathbf{Q}) \equiv \epsilon \pmod{3}$, we set $\hat{\mathbf{P}} = \mathbf{P} + \mathbf{Q}$. We note that, for the associated bilinear form defined by

$$f(\mathbf{X}, \mathbf{Y}) = \sum_{i \leqslant j} a_{ij}\, x_i\, y_j,$$

$$f(\mathbf{P}, \hat{\mathbf{P}}) = f(\mathbf{P}) + f(\mathbf{P}, \mathbf{Q});$$

$$f(\hat{\mathbf{P}}) \equiv f(\mathbf{P}) + 2f(\mathbf{P}, \mathbf{Q}) + \epsilon \pmod{3}.$$

Therefore, for $T \equiv 2^{n-2} \pmod{3}$,

$$f(\mathrm{P} + T2^{n-1}\hat{\mathrm{P}}) = f(\mathrm{P}) + 2^n \; Tf(\mathrm{P}, \hat{\mathrm{P}}) + 2^{2n-2} T^2 f(\hat{\mathrm{P}})$$

$$\equiv f(\mathrm{P}) + 2^n [T(f(\mathrm{P}) + f(\mathrm{P}, \mathrm{Q})) + 2^{n-2}(f(\mathrm{P}) + 2f(\mathrm{P}, \mathrm{Q}) + \epsilon)] \pmod{3 \cdot 2^n}$$

$$\equiv f(\mathrm{P}) + 2^n [2^{n-1} f(\mathrm{P}) + 2^{n-2}\epsilon]$$

$$\equiv \epsilon + 2^n m + 2^n [2^{n-1}(\epsilon + 2^n m) + 2^{n-2}\epsilon]$$

$$\equiv \epsilon + (2^n + 2^{n-2})\, m$$

$$\equiv \epsilon \pmod{3 \cdot 2^n}.$$

Hence, for any $|D| \leqslant 2^{n+2}$ it suffices to shows that each of

$f(\mathrm{X}) \equiv \pm 1 \pmod{2^t}$ is solvable for $2^t \| d$.

By Lemma 3, we write $f \underset{2}{\sim} f_0 + 2g_1 = h$ and consider the choices for f_0. If, for the some $v \geqslant 1$, $f \cong H_v + f_0$, then f_0 properly represents both ± 1. We may thus assume $v = 0$. Also, since f is primitive, f_0 represents some odd integer. Therefore, if $g_1 \cong H_w + g$, for some $w \geqslant 1$, we would obtain that h represents all odd integers. We may thus assume that $v = w = 0$. We write $g_1 \cong f_1 + 2g_2$ and observe that $n(f_0), n(f_1) \leqslant 2$. From Lemma 11, $f_0 \underset{\neq}{\sim} E_2$ and so $2^n(f_0)$ divides det f_0, and $2^{n+n(g_2)}$ divides D. Since $|D| \leqslant 2^{n+2}$, we obtain that $g_2 = H_1, E_2, cx_5^2$, and $|D| = 2^{n+2}$. $f_1 \cong E_2$. Also, since $h \underset{2}{\sim} f$, then det $h \cdot$ det f is a 2-adic square; that is, its odd part is congruent to 1 (mod 8). Letting d_1 be the odd part of det g_2 and recalling that $f_0 = bx_1^2$ or $\pm x_1^2 + bx_2^2$ we thus obtain that

$$bd_1 \equiv \pm 3 \pmod 8. \qquad \ldots(10)$$

We also note that, for $g = -x_1^2 + 2E_2(x_2, x_3)$.

$n(1, 0) = -1$ and $g(-1, 1, 0) = 1$. Hence, we may assume that $b \neq -1$. In Table XIV below we obtain P_1, P_2 such that $h(\mathrm{P}_1) \equiv 1 \pmod d$ and $h(\mathrm{P}_2) \equiv -1 \pmod d$ for the remaining choices of h. The congruence in (10) and $n \geqslant 5$ are used to obtain the possibilities for f_0 given in column 3.

Proposition 3—Let $n \geqslant 4$. If f is an indefinite primitive integral quadratic form with for each $i \neq j$, a_{ij} is even, and $|D| \leqslant 2^{n+2}$, then there exists r, such that $f \sim \pm x_1^2 \pm \ldots \pm x_r^2 \pm h$, for suitable choice of signs, where sum is disjoint and h is given in Table XV

TABLE XIV

g_2	d_1	f_0	n	$\lvert d \rvert$	P_1	P_2
E_2	3	x_1^2	5	2^6	(1, 0)	(1, 2, 1, 2, 2)
		$x_1^2 + x_2^2$	6	2^8	(1, 0)	(1, 0, 5, 3, 5, 2)
H_1	1	$3x_1^2$	5	2^6	(1, 1, 0, −1, 1)	(1, 0, 0, −1, 1)
		$-3x_1^2$	5	2^6	(1, 0, 0, 1, 1)	(1, 1, 0
		$x_1^2 + 3x_2^2$	6	2^8	(1, 0)	(0, 1. 0, 0, −1, 1)
		$x_1^2 - 3x_2^2$	6	2^8	(1, 0)	(0, 1, 1, 0)
$\pm x_5^2$	1	$x_1^2 + 3x_2^2$	5	2^6	(1, 0)	(2, 1, 2, 4, 0)
		$x_1^2 - 3x_2^2$	5	2^6	(1, 0)	(0, 1, 1, 0)
$3x_5^2$	3	$x_1^2 + x_2^2$	5	2^6	(1, 0)	(1, 0, 7, 1, 1)
$-3x_5^2$	3	$x_1^2 + x_2^2$	5	2^6	(1, 0)	(1, 2, 1)

TABLE XV

$2^{-n} \lvert D \rvert$	h
1	0
2	$2E_1$
3	$2E_2$; $3E_6$
4	$2(E_1 \pm E_1)$; $4E_1$; $2E$; $4H_1$; $2E_4$; $2E_5$; $2\hat{E}_6$

PROOF : Since, for each $i \neq j$, a_{ij} is even, then 2^n divides D and $\lvert D \rvert \leqslant 2^{n+2}$ implies that $\lvert D \rvert = r \cdot 2^n$, for some $1 \leqslant r \leqslant 4$.

If $n \geq 5$, we can use Lemma 12 repeatedly until we obtain

$$f = \pm x_1^2 \pm \cdots \pm x_t^2 + f_1,$$

for some indefinite integral quadratic form f_1. The process stops when either $n(f_1) = 4$ or f_1 is not primitive. We set $n_1 = n(f_1)$ and $D_1 = \det f_1$.

We first consider the case when f_1 is not primitive and $n_1 \geqslant 4$. Since $\lvert D_1 \rvert = 2^n.r$, then $f_1 = 2f_2$ for some primitive indefinite quadratic from f_2 with $n(f_2) = n_1 \geqslant 4$

and $|\det f_2| = r$. Recalling that Lemma 10 and Prop. 1 were obtained without using the condition $G(f, V) = 1$, we thus have that $f_2 \sim \pm F_v + H_t + g$, for some choice of sign, $v, t \geqslant 0$, and g given in Table IX or X. Recalling that $\pm F_v \equiv H_{4v}$ and

$$(-x_1)(x_2) = -x_1 x_2 \text{ and } (x_1 + x_2 + x_3)^2 - 2(x_1 + x_3)(x_2 + x_3)$$
$$= x_1^2 + x_2^2 - x_3^2$$

we obtain that

$$\pm x_1^2 + 2(\pm F_v + H_t) \sim \pm x_1^2 \pm \dots \pm x_{2t+1}^2, \text{ for some choice of sign} \quad \dots(11)$$

Thus, for the casewhen f_1 is not primitive, each $h = 2g$ appears in Table XV.

Therefore, it suffices to consider the case when

$$f \sim \pm x_1^2 \pm \dots \pm x_{n-4}^2 + f_1,$$

where f_1 is primitive and $|D_1| = 16r$.

If $r = 3$, we first analyze 3-adic equivalence. From $n_1 = 4$, $3 \| D_1$, and Lemma 3, we have $f_1 \underset{3}{\sim} H_1 + cx_3^2 + 3dx_4^2$. Since

$$H_1(x_1 - x_2, x_1 + x_2) = x_1^2 - x_2^2 \text{ and } cx_3^2 + 3dx_4^2 \equiv \pm x_3 + 3x_4^2 \pmod 3$$

then

$$f \underset{3}{\sim} x_1^2 - x_2^2 \pm x_3^2 + 3x_4^2 ;$$

that is, there is exactly one 3-adic equivalence class for each of $D_1 = 48$, $D_1 = -48$. $D_1 = -48$. Hence, for $r = 3$ (and so for all $1 \leqslant r \leqslant 4$), congruential equivalence follows from 2-adic equivalence. We write $f_1 \underset{2}{\sim} g_0 + 2g_1$; Lemma 11 implies that

$g_0 = cx_1^2$ or $g_0 = \pm x_1^2 + cx_2^2$.

If $r = 1$, then $\det g_1$ is odd and so $g_1 = E_2$ or $g_1 = H_1$. Since $n_1 = 4$, by Lemma 4 either $f_1 \underset{2}{\sim} \pm x_1^2 \pm 3x_2^2 + 2E_2$ or $f_1 \underset{2}{\sim} \pm x_1^2 + x_2^2 + 2H_1$, with the latter equivalent to $\pm x_1^2 \pm \dots \pm x_4^2$ by (11). We note that $3(x_1 + x_2)^2 - (x_1 + x_2 + x_3)^2 - (x_1 + x_3)^2 \equiv x_1^2 \pm 2E_2 \pmod 4$ and so Lemma 4 implies that

$$x_1^2 \pm 2E_2 \underset{2}{\sim} 3x_1^2 - x_2^2 - x_3^2 . \quad \dots(12)$$

Since $\pm\ 3x_1^2 \pm 3x_2^2 \equiv \mp\ x_1^2 \mp x_2^2$ (mod 4), again using

Lemma 4 we have $\pm\ 3x_1^2 \pm 3x_2^2 \underset{2}{\sim} \pm\ x_1^2 \mp x_2^2$. Combining these results, $\pm\ x_1^2 \pm 3x_2^2 + 2E_2 \underset{2}{\sim}$

$$\pm\ x_1^2 \pm x_2^2 \pm 3x_3^2 \pm 3x_4^2 \underset{2}{\sim} \pm\ x_1^2 \pm x_2^2 \pm x_3^2 \pm x_4^2;$$

that is, $h = 0$ when $r = 1$.

In a similar manner, if $r = 3$ then either $f_1 \underset{2}{\sim} \pm\ x_1^2 \pm x_2^2 + 2E_2$

or $f_1 \underset{2}{\sim} \pm x_1^2 \pm 3x_2^2 + 2H_1$. Again using (11), (12), $\pm\ 3 = \mp\ 1$ (mod 4), and Lemma 4, an analysis of disjoint summands yields $h = 2E_2$ when $r = 3$.

Recalling that $f_1 \underset{2}{\sim} g_0 + 2g_1$ with $g_0 = cx_1^2$ or $\pm\ x_1^2 + cx_2$,

for $r = 2$ we have $2 \| \det g_1$. From $n\,(g_1) \geqslant 2$ we thus obtain $g_1 = H_1 + bE_1$. Hence, $f_1 \underset{2}{\sim} cx_1^2 + 2\,(x_2\,x_3 + bx_4^2)$. By considering $-f$, if necessary, we may assume that $c = 1, 3$. Also, since $\pm\ cb$ is a 2-adic square, we may assume that $b = \pm\ c$. If $c = 1$, then (11) implies that $f_1 \sim x_1^2 + x_2^2 - x_3^2 \pm 2x_4^2$; $h = 2E_1$. For $c = 3$, we consider the cases $b = \pm\ 3$ separately : From

$$(x_1 + 2x_2)^2 + 2\,(x_1 + x_2)^2 \equiv 3\,(x_1^2 + 2x_2^2)\ (\text{mod } 8)$$

and Lemma 4 we obtain $3\,(x_1^2 + 2x_2^2) \underset{2}{\sim} x_1^2 + 2x_2^2$.

Hence, using (11),

$$3\,(x_1^2 + 2x_4^2) + 2H_1 \underset{2}{\sim} x_1^2 + 2x_4^2 + 2H_1 \sim x_1^2 + x_2^2 - x_3^2 + 2x_4^2;$$

$h = 2E_1$. Finally, for $b = -3$, we observe that

$$3\,(x_1 + x_2 + x_3)^2 + 2\,(-x_1 - x_3)\,(x_2 + x_1)$$

$$\equiv -x^2 - x_2^2 - 3x_3^2 \ (\text{mod } 4);$$

$$3E_1 + 2H_1 \underset{2}{\sim} -x_1^2 - x_2^2 - 3x_3^2 \text{ , by Lemma 4.}$$

Therefore,

$$3x_1^2 + 2H_1 - 6x_4^2 \underset{2}{\sim} -x_1^2 - x_2^2 - 3(x_3^2 + 2x_4^2)$$

as above, and $h = -2E_1$,

For $r = 4$, we again write $f_1 \underset{2}{\sim} g_0 + 2g_1$, where $g_0 = cx_1^2$ or $\pm x_1^2 + cx_2^2$; $4 \| \det g_1$. Considering $-f$, if necessary, we may assume that $c = 1, 3$ and obtain the possibilities given in Table XVI, (using the determinant argument from Lemma 4). The empty entries of column 3 are discussed below.

TABLE XVI

g_0	g_1	$f_1 \underset{2}{\sim}$
x_1^2	$H_1 \pm 2x_4^2$	$x_1^2 \pm x_2^2 - x_3^2 \pm 4x_4^2$, by (11)
$3x_1^2$	$H_1 \pm 6x_4^2$	
$\pm x_1^2 + x_2^2$	$\pm x_3^2 \pm x_4^2$	$\pm x_1^2 \pm x_2^2 \pm 2x_3^2 \pm 2x_4^2$
$\pm x_1^2 + 3x_2^2$	$\pm x_3^2 \pm 3x_4^2$	
$\pm x_1^2 + cx_2^2$	$2H_1$; $c = 1$	$\pm x_1^2 + x_2^2 + 4H_1$
	$2E_2$; $c = 3$	

Each entry in column 3 of Table XVI is listed in Table XV. Noting that

$$3x_1^2 \pm 12x_4^2 \equiv 3x_1^2 \mp 4x_1^2 \pmod{16}$$

and

$$3x_1^2 + 2H_1 \equiv -x_1^2 + 2H_1 \pmod 4$$

we obtain

$$\begin{aligned} 3x_1^2 + 2H_1 \pm 12x_4^2 &\underset{2}{\sim} 3x_1^2 + 2H_1 \mp 4x_4^2 \\ &\sim -x_1^2 + 2H_1 \mp 4x_4^2 \\ &\underset{2}{\sim} x_1^2 + x_2^2 - x_3^2 \mp 4x_4^2 , \end{aligned}$$

the latter by (11). It thus suffices to consider

$$f_1 \underset{2}{\sim} \pm\, x_1^2 + 3x_2^2 \pm 2x_3^2 \pm 6x_4^2$$

and

$$f_1 \underset{2}{\sim} \pm\, x_1^2 + 3x_2^2 + 4E_2.$$

We note that

$$\pm\,(3x_2)^2 + 3\,(3x_1 + 4x_2 + 2x_3)^2 + 6\,(x_1 + 2x_2 + x_3)^2$$

$$\equiv x_1^2 \pm x_2^2 + 2x_3^2 \pmod 8;$$

$$2x_1 + x_2 + 2x_3)^2 + 3\,(x_1 + 2x_2)^2 - 6\,(2x_1 + 3x_2 + x_3)^2$$

$$\equiv -x_1^2 - x_2^2 - 2x_3^2 \pmod 8;$$

$$-\,(x_1 + 2x_2 + 2x_3)^2 + 3\,(x_2 + 2x_3)^2 - 6\,(x_1 + x_2 + x_3)^2$$

$$\equiv x_1^2 + x_2^2 + 2x_3^2 \pmod 8$$

and also that the determinant of each transformation is odd. Hence, by Lemma 4.

$$\pm\, x_1^2 + 3x_2^2 \pm 6x_4^2 \underset{2}{\sim} \pm\, x_n^2 \pm 2x_4^2\ x_1^2$$

for some choice of signs and so the fourth row of Table XVI has $h = 2\,(E_1 \pm E_1)$.

Moreover,

$$3\,(x_1 + 2x_2 - 2x_3)^2 + 4E_2\,(2x_1 + x_2 - x_3,\ x_1 + x_3)$$

$$\equiv -x_1^2 + 4x_2\,x_3 \pmod{16};$$

$$-3\,(x_1 + 2x_2 + 2x_3)^2 + 4E_2\,(-x_2 + x_3,\ x_1 + 2x_2 + x_3)$$

$$\equiv x_1^2 + 4x_2\,x_3 \pmod{16};$$

where again each determinant is odd. By Lemma 4, $\pm\, 3x_1^2 + 4E_2 \underset{2}{\sim} \mp\, x_1^2 + 4H_1$ and the sixth row of Table XVI satisfies the conclusion with $h = 4H_1$.

Finally we consider the second row; namely, $f_1 \underset{2}{\sim} 3\,(x_1^2 \pm 4x_4)^2 + 2H_1$. We note that $3\,((x_1 + 4x_4)^2 + 4\,(x_1 + x_4)^2) \equiv -\, x_1^2 - 4x_4^2 \pmod{16}$;

$$3\,((3x_1 + 4x_4)^2 - 4\,(x_1 + x_4)^2) \equiv -x_1^2 + 4x_4^2 \pmod{16}.$$

Since each of these determinats is odd, $3\,(x_1 \pm 4x_4)^2 \underset{2}{\sim} -x_1^2 \mp 4x_4^2$.

Hence, by (11),

$$f_1 \underset{2}{\sim} -x_1^2 + 2H_1\ 4x_4^2 - x_1^2 - x_2^2 + x_3^2 \mp 4x_4^2\,;\ \ h = 4E_1.$$

Summarizing, we are considering f for which $f \sim \pm x_1^2 \pm \ldots \pm x_{n-4}^2 + f_1$ for some choice of signs, where f_1 is indefinite, $n\,(f_1) = 4$, and $|\det(f_1)| = r \cdot 2^n$, for some $r \leqslant 4$. We have shown that there exists a choice of signs such that $f_1 \cong \pm x_{n-3}^2 \pm x_{n-2}^2 \pm g$, where g is given in Table XVII.

TABLE XVII

r	1	2	3	4
g	0	$2E_1$	$2E_2$	$4E_1$; $2\,(E_1 \pm E_1)$; $4H_1$

Setting $h_1 = \pm x_{n-3}^2 \pm x_{n-2}^2 \pm g$, we thus have that $f_1 \equiv h$ and so $s\,(f_1) \equiv s\,(h_2)$ (mod 8). Since f_1 is indefinite and $n\,(f_1) = 4$, then $|s\,(f_1)| \leqslant 3$ and so $s\,(f_1) = s\,(h_1)$. Lemma 2 thus implies that $f_1 \sim h_1$, which completes the proof of Proposition 3.

For convenience, we set $K_{t_1,t_2} = -x_1^2 - \ldots - 2x_{t_1}^2 + x_{t_1+1}^2 + \ldots + x_{t_1+t_2}^2$.

Proposition 4—Let f be an indefinite, primitive, integral quadratic form of rank $n \geqslant 21$. If $V \in \mathbb{R}^n$ for which $G\,(f, V)\ n/|D| > \frac{1}{6}$, and $G\,(f, V) = 2$, then $f - K_{t_1,t_2} \pm h$, for some h given in Table XV. Moreover, under this equivalence, for all $1 \leqslant i \leqslant t_1 + t_2$, $v_i \equiv \frac{1}{2}$ (mod 1). Also,

$$\{K_{t_1,t_2}\,(X + V) : X \text{ is integral}\} = K_{t_1,t_2}\,(1/2) + 2\,\mathbb{Z}. \qquad \ldots(13)$$

PROOF : From Lemma 6 we know that $f \equiv L^2$ (mod 2) and so in particular, for $i \neq j$, a_{ij} is even. Hence, we obtain the first conclusion from Proposition 3. By the disjointness of the sum it suffices to complete the proof for $f = K_{t_1,t_2}$. For any i, we take $0 \leqslant v_i < 1$. Again by disjointness, $G\,(f, V) = 2$ implies that $(1 \pm v_i)^2 - v_i^2$ is zero or is at least 2; that is, $v_i = \frac{1}{2}$. The set equality (13) follows from Theorem 2 (*i*) in Watson[32], completing the proof of Prop. 4.

TABLE XVIII

$2-n \mid D \mid$	$\pm h$	V_2 (mod 1)	$\pm h\,(V_2)$ (mod 2)	if $P^+(f, V) = 2$ $-t_1+t_2$ (mod 8)	$s(f)$ (mod 8)
1	0	——	0	0	0
2	$2E_1$	$\frac{u}{2}$	$0;\ u=0\ (2)$	0	± 1
			$\frac{1}{2};\ u=1\ (2)$	∓ 2	∓ 1
3	$2E_2$	$\frac{u}{3}(1, 1)$	$\frac{2}{3};\ u=\pm 1\ (3)$	none	none
			$0;\ u=0\ (3)$	0	± 2
	$2E_6$	$\frac{u}{3}(0, 1, 1, 1, 0, -1)$	$-\frac{2}{3};\ u=\pm 1\ (3)$	none	none
			$0;\ u=0\ (3)$	0	∓ 2
4	$2(E_1+E_1)$	$\frac{1}{2}(u_1, u_2)$	$0;\ u_1, u_2=0\ (2)$	0	± 2
			$1;\ u_1, u_2=1\ (2)$	∓ 4	∓ 2
			$\frac{1}{2};\ u \not\equiv u_2\ (2)$	∓ 2	0
	$2(E_1 - E_1)$	$\frac{1}{2}(u_1, u_2)$	$0;\ u_1=u_2\ (2)$	0	0
			$-\frac{1}{2};\ u_1=0,\ u_2=1\ (2)$	± 2	± 2
			$\frac{1}{2};\ u_1=1,\ u_2=0\ (2)$	∓ 2	∓ 2
	$4E_1$	$\frac{u}{4}$	$0;\ u=0\ (4)$	0	± 1
			$\frac{1}{4};\ u=\pm 1\ (4)$	∓ 1	0
			$1;\ u=2\ (4)$	∓ 4	∓ 3
	$2E_3$	$\frac{u}{4}(1, 1, 1)$	$0;\ u=0\ (4)$	0	± 3
			$\frac{3}{4};\ u=\pm 1\ (4)$	∓ 3	0
			$1;\ u=2\ (4)$	∓ 4	∓ 1
	$4H_1$	$\frac{1}{2}(u_1, u_2)$	$0;\ u_1\ u_2=0\ (2)$	0	0
			$1;\ u_1\ u_2=1\ (2)$	∓ 4	∓ 4
	$2E_4$	$\frac{1}{2}(u+v, u, v, 0)$	$1;\ u\neq v\ (2)$	∓ 4	0
			$0;\ u=v=0\ (2)$	0	± 4
			$1;\ u=v=1\ (2)$	∓ 4	0
	$2E_5$	$\frac{u}{4}(1, 1, 2, 0, 2)$	$0;\ u=0\ (4)$	0	∓ 3
			$\frac{5}{4};\ u=\pm 1\ (4)$	± 3	0
			$1;\ u=2\ (4)$	∓ 4	± 1
	$2\hat{E}_6$	$\frac{1}{2}(0, 0, u, v, u+v, u+v$	$0;\ u=v=0\ (2)$	0	∓ 2
			$-\frac{1}{2};\ u\neq v\ (2)$	∓ 2	± 4
			$1;\ u=v=1\ (2)$	∓ 4	± 2

TABLE XIX

$s(f)$ (mod 8)	h	V_2 (mod 1)	$P^+(f, \mathrm{V})n/\mid D\mid$
0	0	— —	1
	$\pm 2(E_1+E_1)$	$\frac{1}{2}(u_1, u_2);\ u_1 \neq u_2\ (2)$	1/4
	$\pm 2(E_1-E_1)$	$\frac{1}{2}(u_1, u_2);\ u_1 = u_2\ (2)$	
	$\pm 4E_1$	$\pm\frac{1}{4}$	
	$\pm 2E_3$	$\pm\frac{1}{4}(1, 1, 1)$	
	$4H_1$	$\frac{1}{2}(u_1, u_2);\ u_1\ u_2 = 0\ (2)$	
	$2E_4$	$\frac{1}{2}(1, u, v, 0);\ u \neq v\ (2)$	
		$\frac{1}{2}(0, 1, 1, 0)$	
	$\pm 2E_5$	$\pm\frac{1}{4}(1, 1, 2, 0\ 2)$	
± 1	$\pm 2E_1$	0	1/2
	$\mp 2E_1$	$\frac{1}{2}$	
	$\pm 4E_1$	0	1/4
	$\mp 2E_3$	$\frac{1}{2}(1, 1, 1)$	
	$\pm 2E_5$	$\frac{1}{2}(1, 1, 0, 0, 0)$	
± 2	$\pm 2E_2$	0	1/3
	$\mp 2E_6$	0	
	$\pm 2(E_1+E_1)$	0	1/4
± 2	$\mp 2(E_1+E_1)$	$\frac{1}{2}(1, 1)$	1/4
	$\pm 2(E_1-E_1)$	$\frac{1}{2}(0, 1)$	
	$\mp 2\hat{E}_6$	0	
	$\pm 2\hat{E}_6$	$\frac{1}{2}(0, 0, 1, 1, 0, 0)$	
± 3	$\mp 4E_1$	$\frac{1}{2}$	1/4
	$\pm 2E_3$	0	
	$\mp 2E_5$	0	
4	$4H_1$	$\frac{1}{2}(1, 1)$	1/4
	$2E_4$	0	
	$2\hat{E}_6$	$1\ (0, 0, u, v, 1, 1);\ u \neq v\ (2)$	

TABLE XX

Consider the case when $P^+(f, \mathrm{V}) < 2$		
$\pm h$	$P^+(f, \mathrm{V}) \leqslant$	$P^+(f, \mathrm{V})n/\mid D\mid \leqslant$
$2E_2;\ 2E_6$	$\frac{23}{12}$	$\frac{1}{3}\left(\frac{23}{24}\right)^n$
Otherwise	$\frac{7}{4}$	$\frac{1}{2}\left(\frac{7}{8}\right)^n$

Proof of the theorems for the case $G(f, V) = 2$

If $P^+(f, V)^n/|D| > \frac{1}{8}$ holds, then $G(f, V)^n/|D| > \frac{1}{8}$. From $G(f, V) = 2$, we obtain $f \sim K_{t_1,t_2} + h$, for some choice of t_1, t_2 and for some h listed in Table XV. Moreover, separating $V = (V_1, V_2)$ as in the sum $K_{t_1,t_2} + h$, by Prop. 4 we have $V_1 = (1/2) \pmod 1$. Also, from the disjointness of $K_{t_1,t_2} + h$, Lemma 6 implies that if $M(X)$ is the linear part of $h(X + V_2)$ then M is integral and $M^2(X) \equiv h(X) \pmod 2$. Each h given in Table XV has even coefficients. Hence $M^2(X) \equiv h\,X \pmod 2$ is equivalent to reguiring that each coefflcient in M is even. Since $h = 2g$, for some g ia Table XII, we thus obtain the first three columns of Table XVIII from Table XII. Column 4 is computed directly from the preceding columns. For the remainder of Table XVIII. we recall (13), obtaining

$$f(V + X) \equiv K_{t_1,t_2}(1/2) + h\,V_2 \pmod 2 \equiv \tfrac{1}{4}(-t_1 + t_2) + h(V_2)$$

$$+ h(V_2) \pmod 2.$$

Hence, $P^+(f, V) = 2$ exactly when $-t_1 + t_2 \equiv -4h(V_2) \pmod 8$. For the last column we use $s(f) \equiv -t_1 + t_2 + s(\pm h) \pmod 8$.

Summarizing, if $P^+(f, V)^n/|D| > \frac{1}{8}$, and $P^+(f, V) = 2$, then we have $f \sim K_{t_1,t_2} + h$ where h and $V = (0, V_2)$ are given below (Tables XIX and XX).

Since $n \geqslant 21$, each of the entries in the third column is less then $\frac{1}{8}$. This completes the proofs of Theorems 1 and 2

For Theorem 3 we note that $P(f, V)^n/|D| > 1/5$ implies that $P^+(f\,V^n/|D| > 1/5$. Therefore, when $P(f,V)^n/|D| > 1/5$ and $G(f, V) = 2$, the above argument applies and so $P^+(f, V) = 2$. Hence, f and V are given in Table XIX. But, for any of these we have $P(f, V) = 0$ and so $P(f, V)^n/|D| \leqslant 1/5$ for all f, V.

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ON COMMON FIXED POINTS IN METRIC SPACES

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Some fixed point theorems for certain contractive type mapping are presented in this note.

Throughout this paper (X, d) will denote a complete metric space unless otherwise stated and R^+, the set of non-negative reals. Recently Kiventidis[1] proved the following :

*Theorem TK*1—Let T be a self-mapping of X such that

$$d(Tx, Ty) \leqslant d(x, y) - W(d(x, y)) \;\forall\; x, y \text{ in } X \qquad \ldots(1)$$

where $W : R^+ \to R^+$ is a continuous function such that $0 < W(r) < r$ for all $r \in R^+ - \{0\}$

Then T has a unique fixed point :

In what follows first we prove a theorem which gives Theorem *TK* 1 as a special case.

Theorem 1—Let T be a continus mapping and T_1, T_2 be any other two mappings of X into itself such that

$$\begin{aligned} &TT_i = T_i T \; (i = 1, 2) \\ &\bigcup_{i=1}^{2} T_i(X) \subseteq T(X) \end{aligned} \qquad \ldots(2)$$

and

$$\begin{aligned} &\text{for all } x, y \text{ in } X \\ &d(T_1 x, T_2 y) \leqslant d(Tx, Ty) - W(d(Tx, Ty)) \end{aligned} \qquad \ldots(3)$$

where $W : R^+ \to R^+$ is a continuous function, with

$$0 < W(r) < r \text{ for all } r \in R^+ - \{0\}.$$

Then $F_{T,T_1,T_2} = \{x \in X : x = Tx = T_1 x = T_2 x\}$ is non-empty. Furthermore $F_{T_1} = F_{T_2} = F_{T,T_1,T_2} = \{u\}$, for some u in X.

Proof : Let x_0 be an arbitrary point in X.

Since $T_1(X)$ and $T_2(X)$ are subsets of $T(X)$, we let $T_1 x_{2n} = Tx_{2n+1}$ and $T_2 x_{2n+1} = Tx_{2n+2}$, $n = 0, 1, 2, \ldots$.

Then from (3) we have for all $n \geqslant 1$, $x \in X$,

$$\sum_{r=0}^{n} w(d(Tx_r, Tx_{r+1})) \leqslant d(Tx_0, Tx_1).$$

So the series of non-negative terms

$$\sum_{r=0}^{n} W(d(Tx_r, Tx_{r+1})) \text{ is convergent.}$$

From this it follows that $\lim\limits_{r\to\infty} W(d(Tx_r, Tx_{r+1})) = 0$.

Since $W(0) = 0$, so from the continuity of W we get

$$\lim_{r\to\infty} W(d(Tx_r, Tx_{r+1})) = 0$$

$$\Rightarrow W(\lim_{r\to\infty} d(Tx_r, Tx_{r+1})) = 0$$

$$\Rightarrow \lim_{r\to\infty} d(Tx_r, Tx_{r+1}) = 0$$

which implies that $\{Tx_n\}$ is Cauchy and so it converges to a point u in X, since X is complete.

Therefore $\{Tx_{2n+1} = T_1 x_{2n}\}$, $\{Tx_{2n+2} = T_2 x_{2n+1}\}$ and $\{Tx_{2n} = T_2 x_{2n-1}\}$ being subsequences of $\{Tx_n\}$ converge to u also. But $TT_i = T_i T$, $i = 1, 2$ and the continuity of T implies that $\lim\limits_{n\to\infty} T(Tx_{2n}) = Tu$, $\lim\limits_{n\to\infty} T(Tx_{2n+1}) = Tu$, $\lim\limits_{n\to\infty} T_1(Tx_{2n}) = \lim\limits_{n\to\infty} T(T_1 x_{2n}) = Tu$ and $\lim\limits_{n\to\infty} T_2(Tx_{2n+1}) = \lim\limits_{n\to\infty} T(T_2 x_{2n+1}) = Tu$.

Now from (3)

$$d(T_1(Tx_{2n}), T_2 u) \leqslant d(T(Tx_{2n}), Tu) - W(d(T(Tx_{2n}), Tu)$$

Proceeding to the limit $n \to \infty$, we obtain $Tu = T_2 u$.

In a similar manner we can show that $Tu = T_1 u$. Suppose $u \neq Tu$

Now

$$d(T_1(Tx_{2n}), T_2 x_{2n+1})$$
$$\leqslant d(T(Tx_{2n}), Tx_{2n+1}) - W(d(T(Tx_{2n})), Tx_{2n+1}).$$

Proceeding to the limit $n \to \infty$, we obtain

$$d(Tu, u) \leqslant d(Tu, u) - W(d(Tu, u)),$$

which is a contradiction. Thus $u = Tu$.

So $u = Tu = T_1 u = T_2 u$. So F_{T, T_1, T_2} is nonempty. It follows easily from (3)

$$F_{T_1} = F_{T_2} = F_{T, T_1, T_2} = \{u\}.$$

Remarks : Theorem *TK* 1 follows from Theorem 1 is one takes $T_1 = T_2$ and $T = I_X$ where I_X is the identity mapping on X.

In what follows we don't take X as a complete metric space.

Theorem 2—Let T be continuous mapping of a metric space X into itself satisfying (1). If there exists a subset M of X and a point x_0 in M such that

$$d(x, x_0) - d(Tx, Tx_0) \geqslant 2d(x_0, Tx_0) \text{ for every } x \text{ in } x - M \qquad \ldots(4)$$

and if T maps M into a compact subset of X then there exists a unique fixed point of T.

Proof : Since T maps M into a compact set, Theorem 2 will follow from Theorem *TK* 1 if it is shown that $x_n \in M$ for every n, where $x_n = T^n x_0$, $n = 1,2,3,\ldots$ Let us suppose that $x_0 \neq Tx_0$. Then it follows easily that the sequence $\{C_n\}$, where $C_n = d(x_n, x_{n+1})$, is non-increasing and since $x_0 \neq Tx_0$, we get $d(x_n, x_{n+1}) < d(x_0, Tx_0)$.

But

$$d(x_n, x_0) \leqslant d(x_n, x_{n+1}) + d(Tx_n, Tx_0).$$

So

$$d(x_n, x_0) - d(Tx_n, Tx_0) \leqslant d(x_n, x_{n+1}) + d(x_0, Tx_0) < 2d(x_0, Tx_0).$$

Hence it follows from (4) that $x_n \in M$ for every n.

This completes the proof of Theorem 2.

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ON THE STABILITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH COMPLEX COEFFICIENTS

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Necessary and sufficient conditions for the asymptotic stability of a system of differential equations of dimension at most 4 with complex coefficients are established. When the coefficients are real, these conditions coincide with the well known Routh-Hurwitz conditions which we extend to include stability and uniform stability.

1. INTRODUCTION

Consider the homogeneous, first order linear system of ordinary differential equations of n-dimensions $X' = A X$ where A is an $n \times n$ real or complex constant matrix and $X(t)$ is a column vector of the n dependent variables. The characteristic equation is a polynomial equation of degree n whose roots real or complex are the eigenvalues of A.

We will discuss the stability of the system $X' = A X$ when A is either a 2×2, 3×3 or 4×4 real or complex matrix, and for this we follow the definitions of asymptotic stability, uniform stability and stability as given in Jordan and Smith[6].

First we note that when A is constant, then stability implies uniform stability (remarks following Definition 9.3 of Jordan and Smith[6]).

Theorem 9.3 of Jordan and Smith[6] relates the question of stability to the nature of the eigenvalues of A. Details for the case of complex A can he found in Boyce and Diprima[2]. In case of asymptotic stability, we introduce the notions of Hurwitz polynomials and positive functions which will be defined in the next section and used to establish necessary and sufficient conditions for the asymptotic stability of the system $X' = AX$ where A is a complex matrix of dimension at most 4. When A is real, these conditions will automatically reduce to the well-known Routh-Hurwitz's conditions, which we will extend to include not only asymptotic stability but also stability and therefore uniform stability.

Hurwitz polynomials are intimately connceted with the problem of stability of mechanical systems and electrical networks and have numerous practical applications.

2. Definitions and Notations

Definition 2.1—The polynomial $f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$ with complex coefficients is a Hurwitz polynomial if all its roots have negative real parts.

Definition 2.2—If $g(\lambda)$ is any rational function, its paraconjugate is defined by $g^*(\lambda) = \overline{g(-\bar{\lambda})}$ where $\bar{\lambda}$ denotes the complex conjugate of λ.

When $f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n$

then

$$f^*(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \bar{a}_1 \lambda^{n-1} + \ldots + \bar{a}_{n-2}\lambda^2 - \bar{a}_{n-1}\lambda + \bar{a}_n.$$

Definition 2.3—A function $h(\lambda)$ is said to be positive if $\operatorname{Re} h(\lambda) > 0$ whenever $\operatorname{Re}\lambda > 0$.

Theorem 5.1 of Levinson and Redheffer[7] reduces the study of Hurwitz polynomials to the study of positive functions, since it states that if f and f^* do not vanish simultaneously, then f is a Hurwitz polynomial if and only if the function $h(\lambda) = \dfrac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$ is positive. This function satisfies $h^*(\lambda) = -h(\lambda)$, since $f^{**} = f$ and hence $-h^*$ is positive if h is. Theorem 5.2 of Levinson and Redheffer[7] states that any rational function h such that h and $-h^*$ are both positive can be written in the form :

$$h(\lambda) = a + b\lambda + \frac{b_1}{\lambda - i\omega_1} + \frac{b_2}{\lambda - i\omega_2} + \ldots + \frac{b_n}{\lambda - i\omega_n}$$

where $\operatorname{Re} a = 0$, $b \geqslant 0$, $b_k \geqslant 0$, and where the ω_j are distinct real numbers. According to the proof of this theorem, the ω_j are the roots of $1/h(\lambda)$ and since they are distinct, it is an easy exercise to show that the above expansion of $h(\lambda)$ is unique. The result of this theorem is a systematic decision procedure which will be heavily used in the next section.

3. Asymptotic Stability with Complex A

We need the following :

Lemma 3.1—If $f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a^n$ is a Hurwitz polynomial, then f and f^* cannot have a common root :

Proof : Write $f(\lambda)$ and $f^*(\lambda)$ in the factored forms :

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$$

and

$$f^*(\lambda) = (-1)^n (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2), \ldots (\lambda + \bar{\lambda}_n).$$

Assume that $\lambda_i = -\bar{\lambda}_j$ is a common root of f and f^* for some i and j between 1 and n.

Then $\lambda_i + \bar{\lambda}_j = 0$ which leads to $\text{Re}\,\lambda_i + \text{Re}\,\lambda_j = 0$.
This last relation means that λ_i and λ_j cannot both have negative real parts, which contradicts the fact that f is a Hurwitz polynomial.

Now, we consider the case where A is 2×2 complex matrix with characteristic polynomial

$$f(\lambda) = \lambda^2 + a_1 \lambda + a_2. \text{ Let } f^*(\lambda) = \lambda^2 - \bar{a}_1 \lambda + \bar{a}_2$$

be the paraconjugate of f.

Theorem 3.1—The system $X' = A\,X$ where A is a 2×2 complex matrix with characteristic polynomial $f(\lambda) = \lambda^2 + a_1 \lambda + a_2$ is asymptotically stable if and only if : $\text{Re}\,a_1 > 0$ and $\text{Re}\,a_1 .\ \text{Re}\,(a_1 \bar{a}_2) - (\text{Im}\,a_2)^2 > 0$.

PROOF : $X' = A\,X$ is asymptotically stable if and only if $f(\lambda)$ is a Hurwitz polynomial (Jordan and Smith[6], Theorem 9.3). Assume first that $X' = A\,X$ is asymptotically stable, so by Lemma 3.1. f and f^* cannot have a common root which leads to the fact that

$$h(\lambda) = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$$

is a positive function (Levinson and Redheffer[7], Theorem 5.1).

It is obvious that $h(\lambda)$ can be written in the form :

$$h(\lambda) = \frac{(\text{Re}\,a_1)\,\lambda + i\,(\text{Im}\,a_2)}{\lambda^2 + i\,(\text{Im}\,a_1)\,\lambda + \text{Re}\,a_2}.$$

For any complex number $\alpha \neq 0$, it is easily checked that $\text{Re}\,\alpha$ and $\text{Re}\,(1/\alpha)$ have the same sign. Hence, $1/h$ is positive if h is positive. And since $h^*(\lambda) = -h(\lambda)$, then $(1/h)^* = -1/h$, hence $-(1/h)^*$ is positive. Therefore the function

$$\frac{1}{h(\lambda)} = \frac{\lambda^2 + i\,(\text{Im}\,a_1)\,\lambda + \text{Re}\,a_2}{(\text{Re}\,a_1)\,\lambda + i\,(\text{Im} - a_2)},$$

can be written in the form indicated in Theorem 5.2 of Levinson and Redheffer[7].

We claim that $\text{Re}\,a_1 \neq 0$. For if $\text{Re}\,a_1 = 0$, then

$$\frac{1}{h(\lambda)} = \frac{\lambda^2 + i\,(\text{Im}\,a_1)\,\lambda + \text{Re}\,a_2}{i\,(\text{Im}\,a_2)}$$

where the degree of the numerator exceeds by 2 that of the denominator.

Therefore $1/h(\lambda)$ can no more be written as in Theorem 5.2 of Levinson and Redheffer[7] where the degree of the numerator exceeds that of the denominator by at most one. Hence $\text{Re}\,a_1 \neq 0$.

By executing a long division, we bring $1/h(\lambda)$ to the form :

$$\frac{1}{h(\lambda)} = \frac{i}{(\operatorname{Re} a_1)^2} (\operatorname{Re} a_1 \operatorname{Im} a_1 - \operatorname{Im} a_2) + \frac{1}{\operatorname{Re} a_1} \lambda$$

$$\frac{+\operatorname{Re} a_2 + \dfrac{\operatorname{Im} - a_9}{(\operatorname{Re} a_1)^2} (\operatorname{Re} a_1 \operatorname{Im} a_1 - \operatorname{Im} a_2)}{(\operatorname{Re} a_1)\lambda + i(\operatorname{Im} a^2)} .$$

When we compare this form to that of Theorem 5.2 of Levinson and Redheffer[7] which is unique, we come to the conclusion that $\operatorname{Re} a_1 > 0$ and :

$$\operatorname{Re} a_1 (\operatorname{Re} a_1 \operatorname{Re} a_2 + \operatorname{Im} a_1 \operatorname{Im} a_2) - (\operatorname{Im} a_1)^2 \geqslant 0.$$

The latter inequality can be written as :

$$\operatorname{Re} a_1 . \operatorname{Re} (a_1 \bar{a}_2) - (\operatorname{Im} a_2)^2 \geqslant 0.$$

We claim that $\operatorname{Re} a_1 \operatorname{Re} (a_1 \bar{a}_2) - (\operatorname{Im} a_2)^2 \neq 0$.

for otherwise $1/h(\lambda)$ will have the reduced form :

$$\frac{1}{h(\lambda)} = \frac{i}{(\operatorname{Re} a_1)^2} (\operatorname{Re} a_1 \operatorname{Im} a_1 - \operatorname{Im} a_2) + \frac{1}{\operatorname{Re} a_1} \lambda$$

which means that $f - f^*$ and $f + f^*$ have a common root, which in turns implies that f and f^* have a common root, contradicting the fact that f is a Hurwitz polynomial (Lemma 3.1).

Conversely, if the inequalities of the Theorem are satisfied, then by writing $1/h(\lambda)$ as in the first part of the proof, we conclude that $1/h$ is positive, (Levinson and Redheffer[7],. Theorem 5.2), hence h is positive. It is enough to notice that f and f^* cannot have a common root, for otherwise $f - f^*$ and $f + f^*$ will also have a common root, which implies that the numerator and denominator of $1/h$ have a common factor, hence altering its form.

Hence f is a Hurwitz polynomial (Levinson and Redheffer[7], Theorem 5.1) and the proof is thus complete.

Now, consider the case where A is a 3×3 complex matrix with characteristic polynomial

$$f(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3,$$

and Let

$$f^*(\lambda) = -\lambda^3 + \bar{a}_1 \lambda^2 - \bar{a}_2 \lambda + \bar{a}_3$$ be the paraconjugate of f.

Theorem 3.2—The system $X' = AX$ where A is a 3×3 complex matrix with characteristic polynomial $f(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$, is asymptotically stable it and only it :

(1) $\operatorname{Re} a_1 > 0$

(2) $\operatorname{Re} a_1 . \operatorname{Re}(a_1 \bar{a}_2 - a_3) - (\operatorname{Im} a_2)^2 > 0$

and

(3) $$[\operatorname{Re} a_1 \operatorname{Re}(a_2 \bar{a}_3) - (\operatorname{Re} a_3)^2]. [\operatorname{Re} a_1 . \operatorname{Re}(a_1 \bar{a}_2 - a_3) - (\operatorname{Im} a_2)^2] - [\operatorname{Re} a_1 \operatorname{Im}(\bar{a}_1 a_3) + \operatorname{Re} a_3 \operatorname{Im} a_2]^2]) > 0.$$

PROOF : It is easy to see that

$$h(\lambda) = \frac{\lambda^3 + i(\operatorname{Im} a_1)\lambda^2 + (\operatorname{Re} a_2)\lambda + i(\operatorname{Im} a_3)}{(\operatorname{Re} a_1)\lambda^2 + i(\operatorname{Im} a_2)\lambda + \operatorname{Re} a_3}.$$

If $X' = AX$ is asymptotically stable, then f and f^* cannot have a common root by Lemma 3.1 implying that h is positive. Therefore h can be written as in Theorem 5.2 of Levinson and Redheffer[7].

By a similar argument to that of the previous Theorem, we show that $\operatorname{Re} a_1 \neq 0$. After executing a long division :

$$h(\lambda) = \frac{i}{(\operatorname{Re} a_1)^2}(\operatorname{Re} a_1 . \operatorname{Im} a_1 - \operatorname{Im} a_2) + \frac{1}{\operatorname{Re} a_1}\lambda$$
$$+ \frac{R}{(\operatorname{Re} a_1)\lambda^2 + i(\operatorname{Im} a_2)\lambda + \operatorname{Re} a_3}$$

where

$$R = \frac{1}{(\operatorname{Re} a_1)^2}[\operatorname{Re} a_1 . \operatorname{Re}(a_1 \bar{a}_2 - a_3) - (\operatorname{Im} a_2)^2]\lambda$$
$$+ \frac{i}{(\operatorname{Re} a_1)^2}[\operatorname{Re} a_1 \operatorname{Im}(\bar{a}_1 a_3) + \operatorname{Re} a_3 \operatorname{Im} a_2],$$

(1) is satisfied according to Theorem 5.2 of Levinson and Redheffer[7]. The remarks following this theorem as illustrated by Example 5.1 in Levinson and Redheffer[7] allow us to extract (2) and (3) by repeating the argument of Theorem 3.1. The process is rather lengthy but straightforward and consists of another long division and one further application of Theorem 5.2 of Levinson and Redheffer.

The converse is established along the same lines as in Theorem 3.1, we extend these results a little further.

Theorem 3.3—The system $X' = AX$ where A is a 4×4 complex matrix with characteristic polynomial $f(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$, is asymptotically stable if and only if :

(1) $\operatorname{Re} a_1 > 0.$

(2) $\operatorname{Re} a_1 . \operatorname{Re}(a_1 \bar{a}_2 - a_3) - (\operatorname{Im} a_2)^2 > 0$

(3) $[\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_1 \bar{a}_2 - a_3) - (\mathrm{Im}\, a_2)^2].\ [\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_2 \bar{a}_3 - a_1 \bar{a}_4)$

$- (\mathrm{Re}\, a_3)^2] - [\mathrm{Re}\, a_1 .\ \mathrm{Im}\,(\bar{a}_1 a_3 - a_4) + \mathrm{Re}\, a_3 .\ \mathrm{Im}\, a_2]^2 > 0$

(4) $a .\ b - c^2 > 0$ where :

$$a = [\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_1 \bar{a}_2 - a_3) - (\mathrm{Im}\, a_2)^2].[\mathrm{Re}\ a_1 .\ \mathrm{Re}\,(a_3 \bar{a}_4) - (\mathrm{Im}\, a_4)^2]$$
$$- [\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_1 \bar{a}_4) - \mathrm{Im}\, a_2\ \mathrm{Im}\, a_4]^2 .$$

$$b = [\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_1 \bar{a}_2 - a_3) - (\mathrm{Im}\, a_2)^2].\ [\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_2 \bar{a}_3 - a_1 \bar{a}_4)$$
$$- (\mathrm{Re}\, a_3)^2] - [\mathrm{Re}\, a_1 .\ \mathrm{Im}\,(\bar{a}_1 a_3 - a_4) + \mathrm{Re}\, a_3 .\ \mathrm{Im}\ a_2]^2$$

$$c = [\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_1 \bar{a}_2 - a_3) - (\mathrm{Im}\, a_2)^2].\ [\mathrm{Re}\, a_1 .\ \mathrm{Im}\,(\bar{a}_2 a_4) - \mathrm{Re}\, a_3\ \mathrm{Im}\, a_4].$$
$$+ (\mathrm{Re}\, a_1 .\ \mathrm{Re}\,(a_1 \bar{a}_4) - \mathrm{Im}\, a_2\ \ \mathrm{Im}\, a_4].\ [\mathrm{Re}\, a_1\ \mathrm{Im}\,(\bar{a}_1 a_3 - a_4)$$
$$+ \mathrm{Re}\, a_3 .\ \mathrm{Im}\, a_2].$$

Proof : The procedure is entirely similar to the previous Theorems. $h\ (\lambda)$ can be written in the form :

$$h\,(\lambda) = \frac{(\mathrm{Re}\, a_1)\, \lambda^3 + i\,(\mathrm{Im}\, a_2)\, \lambda^2 + (\mathrm{Re}\, a_3)\, \lambda + i\,(\mathrm{Im}\, a_4)}{\lambda^4 + i\,(\mathrm{Im}\, a_1)\, \lambda^3 + (\mathrm{Re}\, a_2)\, \lambda^2 + i\,(\mathrm{Im}\ a_3\, \lambda + \mathrm{Re}\ a_4)} .$$

In $1/h\,(\lambda)$, the degree of the numerator exceeds that of the denominator by one. After performing a long division, the quotient produces (1) following Theorem 5.2 of Levinson and Redheffer[7]. In this division, the remainder over the divisor represents a second degree polynomial in λ over a third degree polynomial in λ, which when reversed brings us to a position similar to that of Theorem 3.2 and the same argument applies. Again the procedure is lengthy and cumbersome, but straight forward.

4. Stability with Real A

A well known Theorem by *A*. Hurwitz establishes necessary and sufficient conditions for the asymptotic stability of the system $X' = AX$ with real A. (Gopal[5], Theorem 8.5), Different approaches to the proof of this Theorem (sometimes referred to as the Routh-Hurwitz crierion) may be found in Bellman[1], Cremer and Effertz[3] and Fuller[4].

It is straightforward to verify how the results of the previous section, reduce to the Routh-Hurwitz's conditions when A is real. In this section, we extend these conditions to include not only asymptotic stability but also stability and therefore uniform stability.

Theorem 4.1—The system $X' = A\,X$ where A is a 3×3 real matrix whose characteristic polynomial $f\,(\lambda) = \lambda^3 + a_1\, \lambda^2 + a_2\, \lambda + a_3$ has no zero repeated root is stable if and only if

$$a_1 \geqslant 0,\ a_2 > 0,\ a_3 \geqslant 0,\ a_1\, a_2 - a_3 \geqslant 0$$

where asymptotic stability occurs only when all the inequalities are strict.

PROOF : The case of asymptotic stability is settled by Theorem 8.5 of Gopal[5].

For stability which is not asymptotic, $f(\lambda)$ has one of the following forms :

(a) $f(\lambda) = \lambda(\lambda - \alpha)(\lambda - \beta)$

where α and β are real and negative or complex such that Re $\alpha \leqslant 0$ and Re $\beta \leqslant 0$.

(b) $f(\lambda) = (\lambda - \gamma)(\lambda + i\theta)(\lambda - i\theta)$

where γ is real and negative and θ a non-zero real number. (Jordan and Smith[6], Theorem 9.3)

Case (a) : $\lambda(\lambda - \alpha)(\lambda - \beta) = \lambda^3 - (\alpha + \beta)\lambda^2 + \alpha\beta\lambda$

$$= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

Hence,

$$\alpha + \beta = -a_1,\ \alpha\beta = a_2, \text{ and } a_3 = 0.$$

When α and β are real and negative, then $a_1 > 0$ and $a_2 > 0$, when they are complex with zero or negative real parts, then $a_1 \geqslant 0$ and $a_2 > 0$.

Therefore if $f(\lambda)$ has form (a), then :

$a_1 \geqslant 0$, $a_2 > 0$, $a_3 = 0$ and these conditions imply that $a_1 a_2 - a_3 \geqslant 0$. The converse is easily established.

Case (b) : $(\lambda - \gamma)(\lambda + i\theta)(\lambda - i\theta) = \lambda^3 - \gamma\lambda^2 + \theta^2\lambda - \gamma\theta^2$

$$= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

Therefore $a_1 = -\gamma$, $a_2 = \theta^2$ and $a_3 = -\gamma\theta^2$ which leads to :

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0 \text{ and } a_1 a_2 - a_3 = 0.$$

The converse holds, and that ends the proof.

It may be worthwhile for the sake of applications to state explicitely how the variations of the above conditions affect the nature of the eigenvalues of A.

(1) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 > 0$. These are the conditions of nsymptotic stability where the eigenvalues are real and negative or complex with negative real parts.

(2) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1 a_2 - a_3 = 0$, one of the eigenvalues is real and negative, the two others are pure imaginary.

(3) $a_1 > 0$, $a_2 > 0$, $a_3 = 0$, $a_1 a_2 - a_3 > 0$. Zero is a non-repeated eigenvalue. The two others are either real and negative if $a_1^2 - 4a_2 \geqslant 0$ or complex with negative real parts if $a_1^2 - 4a_2 < 0$.

(4) $a_1 = 0, a_2 > 0, a_3 = 3, a_1 a_2 - a_3 = 0$. Zero is an eigenvalue and the two others are pure imaginary.

Theorem 4.2—The system $X' = A X$ where A is a 4×4 real matrix whose characteristic polynomial $f(\lambda) = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4$ has no zero repeated root is stable if and only if :

$$a_1 > 0,\ a_2 > 0,\ a_3 \geqslant 0,\ a_4 \geqslant 0 \text{ and } a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0 \qquad \ldots(1)$$

or,

$$a_1 = 0,\ a_2 > 0,\ a_3 = 0,\ a_4 > 0 \text{ and } a_2^2 - 4 a_4 \geqslant 0 \qquad \ldots(2)$$

where asymptotic stability occurs only when all the inequalities in (1) are strict.

PROOF : Again, Theorem 8.5 of Gopal[5] settles the necessary and sufficient conditions for asymptotic stability which can be simplified to appear as in the statement of our Theorem (Rao[8], Remaks following Theorem 2.4 2).

For stability which is not asymptotic, $f(\lambda)$ has one of the following forms :

(a) $f(\lambda) = (\lambda - \alpha)(\lambda - \beta)(\lambda + i\gamma)(\lambda - i\gamma)$ where α and β are real or complex such that Re $\alpha < 0$ and Re $\beta < 0$, and γ a non-zero real number.

(b) $f(\lambda) = \lambda(\lambda - \alpha)\lambda(\lambda - \beta)(\lambda - \gamma)$ where α is real and negative β and γ are real and negative or complex such that Re $\beta \leqslant 0$ and Re $\gamma \leqslant 0$.

(c) $f(\lambda) = (\lambda - i\alpha)(\lambda + i\alpha)(\lambda - i\beta)(\lambda + i\beta)$ where α and β are non-zero real numbers (Jordan and Smith[6], Theorem 9.3)

Case (a) : $(\lambda - \alpha)(\lambda - \beta)(\lambda + i\gamma)(\lambda - i\gamma)$

$$= \lambda^4 - (\alpha + \beta)\lambda^3 + (\alpha\beta + \gamma^2)\lambda^2 - (\alpha + \beta)\gamma^2 \lambda + \alpha\beta\gamma^2$$

$$= \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4.$$

Hence

$$a_1 = -(\alpha + \beta),\ a_2 = \alpha\beta + \gamma^2,\ a_3 = -(\alpha + \beta)\gamma^2,\ a_4 = \alpha\beta\gamma^2.$$

α and β are real and negative or complex with negative real parts if and only if $\alpha + \beta < 0$ and $\alpha\beta > 0$.

Simple manipulation leads to :

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 > 0 \text{ and } a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 = 0.$$

It is easy to establish the converse.

Case (b) : $\lambda(\lambda - \alpha)(\lambda - \beta)(\lambda - \gamma)$ where α, β and γ satisfy the conditions already stated.

It is obvious that $a_4 = 0$ for $\lambda = 0$ is an eigenvalue.

Therefore $f(\lambda)$ can be written in the form :

$$f(\lambda) = \lambda(\lambda - \alpha)(\lambda - \beta)(\lambda - \gamma) = \lambda(\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3).$$

Theorem 4.1—Implies that $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_1 a_2 - a_3 \geqslant 0$ which when combined with $a_4 = 0$ leads to

$$a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 \geqslant 0.$$

The converse is easily established

Case (c) : $f(\lambda) = (\lambda - i\alpha)(\lambda + i\alpha)(\lambda - i\beta)(\lambda + i\beta) = \lambda^4 + (\alpha^2 + \beta^2)\lambda^2$
$$+ \alpha^2 \beta^2 = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4,$$

therefore

$$a_1 = a_3 = 0,\ a_2 = \alpha^2 + \beta^2,\ a_4 = \alpha^2 \beta^2.$$

α^2 and β^2 are then the roots of the quadratic equation $X^2 - a_2 X + a_4 = 0$ which has real positive roots if and only if $a_2^2 - 4a_4 \geqslant 0$, $a_2 > 0$, $a_4 > 0$.

Therefore $a_1 = 0$, $a_2 > 0$, $a_3 = 0$, $a_4 > 0$ and $a_2^2 - 4a_4 \geqslant 0$.

The converse also holds and that ends the proof.

Again, we end this section by discussing how the variations of the above conditions (1) and (2) affect the nature of the eigenvalues of A.

$$(1)\quad a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 > 0,\ a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0.$$

These are the conditions of asymptotic stability where the eigenvalues are real and negative or complex with negatixe real parts.

$$(2)\quad a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 > 0,\ a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 = 0.$$

Here two eigenvalues are real and negative or complex with negative real parts and two are pure imaginary.

$$(3)\quad a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 = 0,\ a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0.$$

Zero is a non-repeared eigenvalue, the three others are such that one is real and negative and two are either real and negative or complex with negative real parts.

$$(4)\ a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 = 0,\ a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 = 0.$$

Zero is an eigenvalue, one real and negative, and two pure imaginary.

$$(5) \quad a_1 = 0, a_2 > 0, a_3 = 0, a_4 > 0 \text{ and } a_2^2 - 4a_4 \geqslant 0.$$

All eigenvalues are pure imaginary.

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SINGULARLY PERTURBED INITIAL VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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The asymptotic behaviour as $\epsilon \to 0+$ of solutions of the initial value problem

$$\begin{cases} dx/dt = \mathbf{u}\,(t, \mathbf{x}, \mathbf{y}, \epsilon),\ \mathbf{x}\,(a,\epsilon) = \mathbf{A}\,(\epsilon), \\ \epsilon\, dy/dt = \mathbf{v}\,(t, \mathbf{x}, \mathbf{y}, \epsilon),\ \mathbf{y}\,(a, \epsilon) = \mathbf{B}\,(\epsilon), \\ \qquad\qquad a < t \leqslant b < \infty \end{cases}$$

where $\epsilon > 0$ is a small parameter and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{A}$ and $\mathbf{B}$ are doubly infinite dimensional vector functions, is established using the theory of first order differential inequalities in a Banach space. Under appropriate assumptions, asymptotic estimates for solutions of the above initial value problem are constructed in terms of solutions of the reduced problem

$$\begin{cases} p\ \mathbf{x}/dt = \mathbf{u}\,(t, \mathbf{X}, \mathbf{Y}, 0),\ \mathbf{X}\,(a) = \mathbf{A}\,(0), \\ \mathbf{v}\,(t, \mathbf{X}, \mathbf{Y}, 0) = \mathbf{0}. \end{cases}$$

The estimates so obtained contain boundary layer terms which explicitly describe the nature of the nonuniform behaviour of solutions as functions of t and ϵ.

1. Introduction

The initial value problem (IVP) for the scalar differential equation (DE)

$$\epsilon y'' + f\,(t, y, y', \epsilon) = 0,\ t \in J_0 := (a, b],\ b < \infty \qquad \ldots(1.1)$$

subject to the initial conditions (ICs)

$$y\,(a, \epsilon) = A\,(\epsilon),\ y'\,(a, \epsilon) = B\,(\epsilon), \qquad \ldots(1.2)$$

where $\epsilon > 0$ is a small parameter, was considered by Baxley[1], Weinstein and Smith[2], and others. These authors studied asymptotic behaviour of solutions and their derivatives of the above IVP as the small parameter ϵ goes to zero. The results were extended to finite systems of first order ordinary DEs and higher order ordinary DEs by

Ramanujam and Srivastava[3]. In the present paper, using the theory of first order ordinary differential inequalities in a Banach space, we establish the asymptotic behaviour of solutions of the following IVP as $\epsilon \to 0$:

$$\begin{cases} \mathbf{P}' & (\mathbf{x}, \mathbf{y}) := \mathbf{x}' - \mathbf{u}(t, \mathbf{x}, \mathbf{y}, \epsilon) = 0,\ t \in J_0 \\ \mathbf{Q}' & (\mathbf{x}, \mathbf{y}) := \epsilon\, \mathbf{y}' - \mathbf{v}(t, \mathbf{x}, \mathbf{y}, \epsilon) = Q \\ \mathbf{R}' & (\mathbf{x}, \mathbf{y}) := \begin{cases} \mathbf{x}(a, \epsilon) = \mathbf{A}(\epsilon) + O(\epsilon) \\ \mathbf{y}(a, \epsilon) = \mathbf{B}(\epsilon) + O(\epsilon) \end{cases} \end{cases} \quad \text{.. (1.3)}$$

where

$$\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)^T, \quad \mathbf{y} = (\ldots, y_{-1}, y_0, y_1, \ldots)^T$$

$$\mathbf{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)^T, \quad \mathbf{v} = (\ldots, v_{-1}, v_0, v_1, \ldots)^T$$

$$\mathbf{x}' = (\ldots, x'_{-1}, x'_0, x'_1, \ldots)^T, \ \mathbf{y}' = (\ldots, y_{-1}, y_0, y_1, \ldots)^T$$

$$\mathbf{A} = (\ldots, A_{-1}, A_0, A_1, \ldots)^T \ \mathbf{B} = (\ldots, B_{-1}, B_0, B_1, \ldots)^T$$

a prime$'$ denotes a differentiation with respect to t, $\epsilon > 0$ is a small parameter, Z is the set of all integers, $\mathbf{A}(\epsilon) = \mathbf{A}(0) + O(\epsilon)$ means $A_i(\epsilon) = A_i(0) + O(\epsilon)$ with 0 as the Landau order symbol. The consideration of the above IVP (1.3) is motivated by problems arising when method of lines [4,5] is applied to singular perturbation problems for parabolic differential equations with a small parameter ϵ multiplying the time derivative. Also, if one considers the DE (1.1) with the ICs (1.2) in a Banach space of all doubly infinite real sequences, one obtains IVPs of the form given (1.3) using proper change of variables.

In section 2 we state and prove monotonicity theorems in a Banach space which shall be used in the rest of the paper. Sections 3-4 deal with IVPs for second order linear systems defined in the Banach space and obtains explicit bounds for solutions and their derivatives. In turn, these estimates are used to discuss the asymptotic behaviour of solutions and their derivatives. The results of section 3 are generalised to cover IVPs for nonlinear systems in section 5.

2. Preliminaries

In this section monotonicity theorems for first order ordinary DEs defined in a Banach space are developed. The later sections of this paper make use of these theorems in obtaining estimates and hence studying the asymptotic behaviour of solutions of the IVPs mentioned in section 1. The proofs of the present monotonicity theorems are different from that of ones given in Walter[4]. Also, they are generalization of the theorems presented for the ordinary DEs.

Let $B = l^\infty(Z, R)$ be the Banach space of all doubly infinite real sequences with the norm of $\mathbf{u} \in B$ given by

$$\|\mathbf{u}\| = \sup_{i \in Z} |u_i| .$$

If $\mathbf{v}, \mathbf{w} \in B$, then $\mathbf{v} \leqslant \mathbf{w}$ and $\mathbf{v} < \mathbf{w}$ respectively mean $v_i \leqslant w_i$, $i \in Z$ and $v_i \leqslant w_i - \delta$, $i \in Z$, for some positive real number δ.

The continuity and differentiability of functions $\mathbf{x}(t) : J \to B$, can be defined as follows. The function $\mathbf{x}(t)$ is continuous at a point $t_0 \in J$ (respectively differentiable with the derivative $\mathbf{x}'(t_0)$) if

$$\|\mathbf{x}(t_0 + h) - \mathbf{x}(t_0)\| \to 0 \text{ as } h \to 0$$

(respectively $\|(\mathbf{x}(t_0 + h) - x(t_0))/_h - \mathbf{x}'(t_0)\| \to 0$ as $h \to 0$).

In the following $C^k(I)$, for a real interval $I \subset R$, stands for the set of all functions k times continuously differentiable in I.

Definition 2.1—Consider a Banach valued function $\mathbf{g} : J_0 \times D \to B, D \subset B$. Then the function $\mathbf{g} = \mathbf{g}(t, \mathbf{z})$ is quasimonotone increasing in $\mathbf{u}$ if there exists a real positive constant M such that

$$\mathbf{u} \leqslant \mathbf{v} \Longrightarrow \mathbf{g}(t, \mathbf{u}) - \mathbf{g}(t, \mathbf{v}) \leqslant M(\mathbf{v} - \mathbf{u}),\ \mathbf{u}, \mathbf{v} \in B. \qquad \text{...(2 1)}$$

Theorem 2.2—Let $\mathbf{u}$ be a solution of the DE

$$\mathbf{P}\,\mathbf{u} := \epsilon\, \mathbf{u}' - \mathbf{g}(t, \mathbf{u}, \epsilon) = 0,\ t \in J_0 \qquad \text{...(2.2)}$$

that is,

$$P_i\,\mathbf{u} := \epsilon\ u_i' - g_i(t, \mathbf{u}, \epsilon) = 0,\ i \in Z$$

subject to the IC

$$\mathbf{R}\,\mathbf{u} := \mathbf{u}(a, \epsilon) = \mathbf{A}(\epsilon),\ \mathbf{u} \in U = C^1(J_0) \cap C(J),\ J := [a, b]. \qquad \text{...(2.3)}$$

Then for every $\mathbf{v}, \mathbf{w} \in U$, the following implication is true :

$$\begin{cases} \mathbf{P}\,\mathbf{v} \leqslant \mathbf{P}\,\mathbf{u} \leqslant \mathbf{P}\,\mathbf{w}, \text{ that is, } P_i\,\mathbf{v} \leqslant P_i\,\mathbf{u} \leqslant P_i\,\mathbf{w}, \\ \mathbf{R}\,\mathbf{v} \leqslant \mathbf{R}\,\mathbf{u} \leqslant \mathbf{R}\,\mathbf{w} \end{cases} \qquad \text{...(2.4)}$$

$\Longrightarrow$

$$\mathbf{v}(t, \epsilon) \leqslant \mathbf{u}(t, \epsilon) \leqslant \mathbf{w}(t, \epsilon),\ t \in J, \qquad \text{...(2.5)}$$

provided that

(i) $\mathbf{g}(t, \mathbf{z}, \epsilon) : J_0 \times D \times (0, \epsilon_1] \to B$ is quasimonotone increasing in $\mathbf{z}$ in the sense of Definition 2.1;

(ii) there exists a real number $\delta > 0$ and a "test function"

$$\mathbf{s}(t, \epsilon) : J \times (0, \epsilon_1] \to B$$

such that

$$s_i(t, \epsilon) = s_j(t, \epsilon),\ i \neq j$$

$$s_i(t, \epsilon) > 0 \text{ on } J, i, j \in Z, \mathbf{s} \in U$$

and

$$\begin{cases} P_i(\mathbf{w} + \alpha_1 \mathbf{s}) - P_i \mathbf{w} \geqslant \alpha_1 \delta > 0 \text{ in } J_0 \\ P_i \mathbf{v} - P_i(v - \alpha_1 \mathbf{s}) \geqslant \alpha_1 \delta > 0, i \in Z \end{cases} \quad \ldots(2.6)$$

for every real positive number α_1 and for every but fixed ϵ. Here $\alpha_1 \mathbf{s} = (\ldots, \alpha_1 s_{-1}, \alpha_1 s_0, \ldots)$;

(iii) there exists a positive constant L such that

$$\mathbf{g}(t, \mathbf{w} + \beta \mathbf{s}, \epsilon) - \mathbf{g}(t_1 \mathbf{w} + \eta \mathbf{s}, \epsilon) \leqslant L \mathbf{s}(t)(\beta - \eta) \quad \ldots(2.7)$$

$$\mathbf{g}(t, \mathbf{v} - \eta \mathbf{s}, \epsilon) - \mathbf{g}(t, \mathbf{v} - \beta \mathbf{s}, \epsilon) \leqslant L \mathbf{s}(t)(\beta - \eta) \quad \ldots(2.8)$$

for every real η, β such that $0 < \eta \leqslant \beta$.

PROOF: In the following, a proof is given to establish the right inequality of (2.5). Similar analysis will yield the left inequality. If the right inequality of (2.5) is not true then there exist $\epsilon_0 > 0, t^* \in J_0$ and $j \in Z$

such that

$$u_j(t^*, \epsilon_0) - w_j(t^*, \epsilon_0) > 0.$$

For this ϵ_0, let

$$P(t, \epsilon_0) = \inf [w_i(t, \epsilon_0) - u_i(t, \epsilon_0), i \in Z], t \in J$$

which is a continuous function of J (Walters[4], p. 98). Hence, there exists a point $t_0 \in J_0$ such that

$$\alpha = -P(t_0, \epsilon_0)/_{s(t_0, \epsilon_0)} = -\min_{t \in J} [P(t, \epsilon_0)/_{s(t, \epsilon_0)}] > 0,$$

where

$$s(t, \epsilon_0) = s_i(t, \epsilon_0).$$

We have

$$\mathbf{u}(t, \epsilon_0) - [\mathbf{w}(t, \epsilon_0) + \alpha \mathbf{s}(t, \epsilon_0)] \leqslant 0. \quad \ldots(2.9)$$

Given $\eta > 0$ such that

$$\eta < \alpha, \eta < \alpha \delta / \max_{t \in J} s(t, \epsilon_0)(L + M + \delta) \quad \ldots(2.10)$$

and hence given $\eta s(t_0, \epsilon_0)$, there exists atleast one $j \in Z$ with the property that

$$u_j(t_0, \epsilon_0) - w_j(t_0, \epsilon_0) > \alpha s(t_0, \epsilon_0) - \eta s(t_0, \epsilon_0) \quad \ldots(2.11)$$

that is,

$$u_j(t_0, \epsilon_0) - [w_j(t_0, \epsilon_0) + (\alpha - \eta) s(t_0, \epsilon_0)] > 0. \quad \ldots(2.12)$$

Hence the function

$$u_j(t, \epsilon_0) - [w_j(t, \epsilon_0) + (\alpha - \eta) s(t, \epsilon_0)]$$

which is negative at $t = a$ and positive at $t = t_0$ attains a positive maximum say at $t = t_1 \in J_0$.

Therefore at $t = t_1$ we have

$$u_j(t_1, \epsilon_0) - [w_j(t_1, \epsilon_0) + (\alpha - \eta) s(t_1, \epsilon_0)] > 0 \qquad \text{...(2.13)}$$

and

$$u'_j(t_1, \epsilon_0) - [w'_j(t_1, \epsilon_0) + (\alpha - \eta) s'(t_1, \epsilon_0)] \geqslant 0. \qquad \text{...(2.14)}$$

Now from (2.6), (2.14), (2.1), (2.7), (2.10) and (2.13) we have at $t = t_1$

$$\begin{aligned}
0 &\geqslant (P_j \mathbf{w} - P_j(\mathbf{w} + (\alpha - \eta)\mathbf{s}))_{t=t_1} + \delta(\alpha - \eta) \\
&\geqslant (P_j \mathbf{v} - P_j(\mathbf{w} + (\alpha - \eta)\mathbf{s}))_{t=t_1} + \delta(\alpha - \eta) \\
&= (v_j - w_j + (\eta - \alpha) s)'(t_1, \epsilon_0) \\
&\quad - (g_j(t_1, \mathbf{v}(t_1, \epsilon_0), \epsilon_0) \\
&\quad - g_j(t_1, \mathbf{w} + (\alpha - \eta)\mathbf{s}, \epsilon_0)) + \delta(\alpha - \eta) \\
&\geqslant -(g_j(t_1, \mathbf{v}, \epsilon_0) - g_j(t_1, \mathbf{w} + \alpha \mathbf{s}, \epsilon_0) \\
&\quad + g_j(t_1, \mathbf{w} + \alpha \mathbf{s}, \epsilon_0) - g_j(t_1, \mathbf{w} + (\alpha - \eta)\mathbf{s}, \epsilon_0)) + \delta(\alpha - \eta) \\
&\geqslant - M((w_j + \alpha s - v_j)(t_1, \epsilon_0)) \\
&\quad - L \eta s(t_1, \epsilon_0) + \delta(\alpha - \eta) \\
&\geqslant - M(\eta s(t_1, \epsilon_0)) - L \eta s(t_1, \epsilon_0) + \delta(\alpha - \eta) > 0.
\end{aligned}$$

It is a contradiction and hence the proof of this theorem.

The following theorem is stated for the system (1.3).

Theorem 2.3— Let $(\mathbf{x}, \mathbf{y})$ be a solution of the IVP (1.3). Then for every $\mathbf{x}, y, \bar{\mathbf{y}} \in U$ the following implication is true

$$\begin{cases}
\mathbf{P}'(\mathbf{x}, \mathbf{y}) \leqslant \mathbf{P}(\mathbf{x}, \mathbf{y}) \leqslant \mathbf{P}'(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \\
\mathbf{Q}'(\mathbf{x}, \mathbf{y}) \leqslant \mathbf{Q}'(\mathbf{x}, \mathbf{y}) \leqslant \mathbf{Q}'(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \\
\mathbf{R}'(\mathbf{x}, \mathbf{y}) \leqslant \mathbf{R}(\mathbf{x}, \mathbf{y}) \leqslant \mathbf{R}'(\bar{\mathbf{x}}, \bar{\mathbf{y}})
\end{cases}$$

$$\mathbf{x} \leqslant \mathbf{x} \leqslant \bar{\mathbf{x}}, \ \mathbf{y} \leqslant \mathbf{y} \leqslant \bar{\mathbf{y}},$$

provided that

(i) $\mathbf{u}(t, \mathbf{p}, \mathbf{q}, \epsilon) : J_0 \times B^2 \times (0, \epsilon_0] \rightarrow B$ is quasimonotone increasing in $\mathbf{p}$ in the sense of Definition 2.1, and monotone increasing in $\mathbf{q}$; $\mathbf{v}(t, \mathbf{p}, \mathbf{q}, \epsilon)$:

$J_0 \times B^2 \times (0, \epsilon] \rightarrow B$ is monotone increasing in $\mathbf{p}$ and quasimonotone increasing in $\mathbf{q}$;

(ii) there exists a positive number δ_1 and *a* "test function" $\mathbf{s}(t, \epsilon) : J \times (0, \epsilon_0] \rightarrow B$ such that

$$s_i(t, \epsilon) = s_j(t, \epsilon) > 0, i, j \in Z, i \neq j, \mathbf{s} \in U$$

and

$$\left\{\begin{array}{l} \mathbf{P}'(\overline{\mathbf{x}} + \alpha_1 \mathbf{s}, \overline{\mathbf{y}} + \alpha_1 \mathbf{s}) - \mathbf{P}'(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \geqslant \alpha_1 \delta_1 > 0, \\ \mathbf{P}^r(\mathbf{x}, \mathbf{y}) - \mathbf{P}'(\mathbf{x} - \alpha_1 \mathbf{s}, \mathbf{y} - \alpha_1 \mathbf{s}) \geqslant \alpha_1 \delta_1 > 0 \\ \mathbf{Q}'(\overline{\mathbf{x}} + \alpha_1 \mathbf{s}, \overline{\mathbf{y}} + \alpha_1 \mathbf{s}) + \mathbf{Q}'(\overline{\mathbf{x}, \mathbf{y}}) \geqslant \alpha_1 \delta_1 > 0 \\ \mathbf{Q}'(x, y) - \mathbf{Q}'(\mathbf{x} - \alpha_1 \mathbf{s}, \mathbf{y} - \alpha_1 \mathbf{s}) \geqslant \alpha_1 \delta_1 > 0 \end{array}\right. \quad \ldots(2.15)$$

for every positive real α_1 and for every but fixed ϵ;

(iii) there exists a positive constant L such that

$$\left\{\begin{array}{l} \mathbf{u}(t, \overline{\mathbf{x}} + \boldsymbol{\beta}\mathbf{s}, \overline{\mathbf{y}} + \beta \mathbf{s}, \epsilon) - \mathbf{u}(t, \overline{\mathbf{x}} + \eta \mathbf{s}, \overline{\mathbf{y}} + \eta \mathbf{s}, \epsilon) \\ \qquad \leqslant L\, \mathbf{s}(t)(\boldsymbol{\beta} - \eta), \\ \mathbf{u}(t, x - \eta \mathbf{s}, y - \eta \mathbf{s}, \epsilon) - \mathbf{u}(t, \mathbf{x} - \beta \mathbf{s}, \mathbf{y} - \boldsymbol{\beta}\mathbf{s}, \epsilon) \\ \qquad \leqslant L\, \mathbf{s}(t)(\boldsymbol{\beta} - \eta). \\ \mathbf{v}(t, \overline{\mathbf{x}} + \beta \mathbf{s}, \overline{\mathbf{y}} + \boldsymbol{\beta}\mathbf{s}, \epsilon) - \mathbf{u}(t, \overline{\mathbf{x}} + \eta \mathbf{s}, \\ \qquad \overline{\mathbf{y}} + \eta \mathbf{s}, \epsilon) \leqslant L\, \mathbf{s}(t)(\boldsymbol{\beta} - \eta) \\ \mathbf{v}(t, x - \eta \mathbf{s}, \mathbf{y} - \eta \mathbf{s}, \epsilon) - \mathbf{u}(t, x - \boldsymbol{\beta}\mathbf{s}, \\ \qquad \mathbf{y} - \beta \mathbf{s}, \epsilon) \leqslant L\, \mathbf{s}(t)(\beta - \eta) \end{array}\right. \quad \ldots(2.16)$$

for every η, β such that $0 < \eta \leqslant \boldsymbol{\beta}$.

PROOF : Under the transformation $z_{2i} = x_i$, $z_{2i+1} = y_i$, $i \in Z$, the system (1.3) reduces to a single doubly infinite system. The hypotheses of the present theorem amount to the same assumptions of Theorem 2.2. Hence the proof of the present theorem.

The following discussion for the linear system illustrates the above Theorem 2.3.

Consider a doubly infinite system of linear second order ordinary DEs

$$\epsilon\, \mathbf{x}'' + \alpha\, \mathbf{x}' + \boldsymbol{\beta}\, \mathbf{x} = \boldsymbol{\gamma}, t \in J_0 \quad \ldots(2.17)$$

subject to the ICs

$$\mathbf{x}(a, \epsilon) = \mathbf{A}(\epsilon), \mathbf{x}'(a, \epsilon) \quad \mathbf{B}(\epsilon) \quad \ldots(2.18)$$

where

$$\alpha = (\alpha_{ij})\ \alpha_{ij} = 0, i \neq j, \ i, j \in Z$$

$$\boldsymbol{\beta} = (\boldsymbol{\beta}_{ij}), \alpha\, \mathbf{x}' = (\ \ , \alpha_{11} x_1', \alpha_{22} x_2', \ldots)^T$$

$$\boldsymbol{\beta}\,\mathbf{x} = (\ldots, \sum_{j=-\infty}^{\infty} \beta_{1j}\, x_j, \sum_{j=-\infty}^{\infty} \beta_{2j}\, x_j, \ldots)^T$$

$$\boldsymbol{\gamma} = (\ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots)^T$$

and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are assumed to be continuous functions in their arguments.

Let $\mathbf{y} := \mathbf{x}'$. Then the above system (2.17) – (2.18) may be written as

$$\begin{cases} P_1(\mathbf{x}, \mathbf{y}) := \mathbf{x}' - \mathbf{y}' = 0 \\ Q_1(\mathbf{x}, \mathbf{y}) := \epsilon\, \mathbf{y}' + \boldsymbol{\alpha}\mathbf{y} + \boldsymbol{\beta}\,\mathbf{x} = 0, \; t \in J_0 \\ R_1(\mathbf{x}, \mathbf{y}) := \begin{cases} \mathbf{x}(a, \epsilon) = \mathbf{A}(\epsilon) \\ \mathbf{y}(a, \epsilon) = \mathbf{B}(\epsilon). \end{cases} \end{cases} \quad \ldots(2.19)$$

Theorem 2.4—The conditions (i) – (iii) of Theorem 2.3 are satisfied for the linear IVP (2.19) provided that

(i) $\boldsymbol{\beta} \leqslant \mathbf{0}$ (ii) $\alpha_{ii} \geqslant \delta_3 > 0, \; i \in Z$

(iii) $\sum_{j=-\infty}^{\infty} \beta_{ij} \geqslant -L, \; L > 0, \; i \in Z.$

PROOF : The assumptions (i) and (ii) of the present theorem yield the condition (i) of Theorem 2.3 whereas the inequalities (2. 16) follow from the condition (iii) of the present theorem. Finally using the assumptions (ii) and (iii) one can show that the function s, $s_i(t, \epsilon) = e^{kt/\epsilon}$ is a required test function, by a proper choise of k, for the system (2.19) satisfying the inequalities (2.15).

3. Estimates and Asymptotic Behaviour of Solutions of Linear Systems—I

In this section we discuss the asymptotic behaviour of solutions and their derivatives of the IVP (2.17) – (2.18) by obtaining necessary estimates. Since the IVPs (2.17)-(2.18) and (2.19) are equivalent in the sense that every solution of one system is a solution of the other and vice versa. We consider only the system (2.19) in the following study.

In the rest of this section it is assumed that all the conditions (i)—(iii) of Theorem 2.4 hold true for the IVP (2.19). Consequently the implication of the Theorem 2.3 is valid for the IVP (2.19). Further, in the following, $M_i = i = 1, 2, \ldots$ are real positive constants independent of ϵ.

Theorem 3.1—Consider the IVP (2.17) - (2.18) and assume that $\mathbf{A}(\epsilon) \equiv \mathbf{O} \equiv \mathbf{B}(\epsilon)$. Then we have

$$\|\mathbf{x}(t, \epsilon)\| \leqslant \|\boldsymbol{\gamma}\|\, e^{m(t-a)} \quad \ldots(3.1)$$

$$\|\mathbf{x}'(t, \epsilon)\| \leqslant m\, \|\boldsymbol{\gamma}\|\, e^{m(t-a)} \quad \ldots(3.2)$$

for some positive constant m and for every solution x of (2.17)—(2.18).

PROOF : Consider the IVP (2.19) and let (x, y) be its solution. Define functions (x, y) and $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ as

$$\bar{x}_i = \|\gamma\|\, e^{m(t-a)},\quad x_i = -\bar{x}_i$$

$$\bar{y}_i = m\,\|\gamma\|\, e^{m(t-a)},\quad y_i = -\bar{y}_i$$

where m is a positive constant independent of ϵ to be chosen suitably.

We have

$$P_{1i}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) := \bar{x}_i' - \bar{y}_i = O = P_{1i}(\mathbf{x}, \mathbf{y})$$

$$Q_{1i}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) := \epsilon\, \bar{y}_i' + \alpha_{ii}\bar{y}_i + \sum_{j=-\infty}^{\infty} \beta_{ij}\bar{x}_j$$

$$= (\epsilon\, m^2 + \alpha_{ii}\, m + \sum_{j=-\infty}^{\infty} \beta_{ij})\, \|\gamma\|\, e^{m(t-a)}$$

$$\geqslant (\epsilon\, m^2 + \delta_3\, m - L)\, \|\gamma\|\, e^{m(t-a)}$$

$$\geqslant \|\gamma\| \geqslant \gamma_i = Q_{1i}(\mathbf{x}, \mathbf{y})$$

by a proper choise of m.

Also

$$\bar{\mathbf{x}}(a, \epsilon) \geqslant \mathbf{O} = \mathbf{x}(a, \epsilon),\ \bar{\mathbf{y}}(a, \epsilon) \geqslant \mathbf{O} = \mathbf{y}(a, \epsilon).$$

Hence, by Theorem 2.3, we have

$$\mathbf{x} \leqslant \bar{\mathbf{x}},\ \mathbf{y} \leqslant \bar{\mathbf{y}}. \qquad \text{...(3.3)}$$

Similar procedure yields

$$x \leqslant \mathbf{x},\ y \leqslant \mathbf{y}. \qquad \text{...(3.4)}$$

Hence

$$|\, x_i(t, \epsilon)\, | \leqslant \|\gamma\|\, e^{m(t-a)} \qquad \text{...(3.5)}$$

and

$$|\, y_i(t, \epsilon)\, | \leqslant m\, \|\gamma\|\, e^{m(t-a)}. \qquad \text{...(3.6)}$$

Taking supremum in (3.5)–(3.6), and using the fact that $\mathbf{y} = \mathbf{x}'$ we get the required estimates (3.1)—(3.2).

Theorem 3.2 Consider the IVP (2.17)—(2.18) and assume that $\gamma \equiv \mathbf{O} \equiv \mathbf{A}$. Then we have

$$\|\mathbf{x}(t, \epsilon)\| \leqslant \epsilon \|\mathbf{B}(\epsilon)\| M_1 [e^{m(t-a)} - e^{-\delta_3 (t-a)/\epsilon}] \qquad \text{...(3.7)}$$

$$\|\mathbf{x}'(t,\varepsilon)\| \leqslant \epsilon \|\mathbf{B}(\epsilon)\| M_2\, e^{m(t-a)} + \|\mathbf{B}(\epsilon)\| e^{-\delta_3(t-a)/\epsilon}] \qquad \ldots(3.8)$$

whers $m > 0$ and $m\,\delta_3 - L \geqslant 0$.

PROOF : Consider the IVP (2.19) and let (x, y) be its solution. Define functions (x, y) and $(\overline{\mathrm{x}}, \overline{\mathrm{y}})$ as follows.

$$\bar{x}_i = \epsilon\, \|\mathbf{B}(\epsilon)\|\, [e^{m(t-a)} - e^{-\delta_3(t-a)/\epsilon}]/(m\,\epsilon + \delta_3)$$

$$\bar{y}_i = \|\mathbf{B}(\epsilon)\|\, [m\,\epsilon\, e^{m(t-a)} + \delta_3\, e^{-\delta_3(t-a)/\epsilon}]/(m\,\epsilon + \delta_3)$$

$$x_i = -\bar{x}_i, \bar{y}_i = -\bar{y}_i, i \in Z.$$

We have

$$P_{1i}(\overline{\mathrm{x}}, \overline{\mathrm{y}}) : = \bar{x}_i' - \bar{y}_i = O = P_{1i}(\mathrm{x}, \mathrm{y})$$

$$Q_{1i}(\overline{\mathrm{x}}, \overline{\mathrm{y}}) : = \bar{y}_i' + \alpha_{1i}\, \bar{y}_1 + \sum_{j=-\infty}^{\infty} \beta_{ij}\, \bar{x}_j$$

$$\geqslant \epsilon\, \|\mathbf{B}(\epsilon)\|\, e^{m(t-a)}\, [m\,\epsilon^2 + m\,\delta_3 - L]/(m\epsilon + \delta_3)$$

$$+ \|\mathbf{B}(\epsilon)\|\, e^{-\delta_3(t-a)/\epsilon}\, [-\delta_3^2 + \delta_3^2 + \epsilon\, L]/(m\epsilon + \delta_3)$$

$$\geqslant O = Q_{1i}(\mathrm{x}, \mathrm{y}), i \in Z.$$

Also

$$\mathrm{x}(a, \epsilon) = \mathbf{O} \leqslant \overline{\mathrm{x}}(a, \epsilon)$$

$$\mathrm{y}(a, \epsilon) = \mathbf{B}(\epsilon) \leqslant \|\mathbf{B}(\epsilon)\| = \overline{\mathrm{y}}(a, \epsilon).$$

Hence, by Theorem 2.3, we have

$$\mathrm{x} \leqslant \overline{\mathrm{x}}, \mathrm{y} \leqslant \overline{\mathrm{y}}. \qquad \ldots(3.9)$$

Similar analysis yields

$$x \leqslant \mathrm{x}, y \leqslant \mathrm{y}. \qquad \ldots(3.10)$$

Hence we have

$$|\, x_i(t, \epsilon)\, | \leqslant \epsilon \|\mathbf{B}(\epsilon)\| M_1\, [e^{m(t-a)} - e^{-\delta_3(t-a)/\epsilon}] \qquad \ldots(3.11)$$

and

$$|\, y_i(t, \epsilon)\, | \leqslant \epsilon\, \|\mathbf{B}(\epsilon)\| M_2\, e^{m(t-a)} + \|\mathbf{B}(\epsilon)\|\, e^{-\delta_3(t-a)/\epsilon} \qquad \ldots(3.12)$$

The estimates (3.7) – (3.8) follow from (3.11)– (3.12).

Theorem 3.3 Let w be a solution of the IVP

$$\alpha\, \mathbf{w}' + \boldsymbol{\beta}\, \mathbf{w} = \gamma, \mathbf{w}(a, \epsilon) = \mathbf{A}(\epsilon) \qquad \ldots(3.13)$$

with the properties that

$$\|w''\| \leqslant M_4 \quad \ldots(3.14)$$

and

$$\lim_{\epsilon \to 0+} w(t, \epsilon) = u(t), \; t \in J \quad \ldots(3.15)$$

where u is the solution of the IVP

$$\alpha(t, 0)\, u' + \beta(t, 0)\, u = \gamma, \; u(a) = A(0). \quad (3.16)$$

If x is a solution of the IVP (2.17)—(2.18) then

$$\|x(t, \epsilon) - u(t)\| \leqslant \epsilon M_3 e^{m(t-a)} + C(\epsilon) + \epsilon M_4 \|B(\epsilon) - u'(a)\| [e^{m(t-a)} - e^{-\delta_3 (t-a)/\epsilon}] \quad \ldots(3.17)$$

$$\|x'(t, \epsilon) - u'(t)\| \leqslant \epsilon M_5 e^{m('-a)} + D(\epsilon) + \epsilon M_6 \|B(\epsilon) - u'(a)\| e^{m(t-a)} + \|B(\epsilon) - u'(a)\| e^{-\delta_3 (n-a)/\epsilon}, \; t \in D \quad \ldots(3.18)$$

where $C(\epsilon)$, $D(\epsilon)$ approach zero as $\epsilon \to 0+$

Consequently

$$\lim_{\epsilon \to 0+} x(t, \epsilon) = u(t), \; t \in J \quad \ldots(3.19)$$

and

$$\lim_{\epsilon \to 0+} x'(t, \epsilon) = u'(t), \; t \in J_0. \quad \ldots(3.20)$$

PROOF :

Let v be the solution of the IVP

$$\begin{cases} \epsilon v'' + \alpha v' + \beta v = \gamma + \epsilon w'', \; t \in J_0, \\ v(a, \epsilon) = A(\epsilon), \; v'(a, \epsilon) = B(\epsilon). \end{cases} \quad \ldots(3.21)$$

From Theorem 3.1 we have

$$\|x(t, \epsilon) - v(t, \epsilon)\| \leqslant \epsilon M_4 e^{m(t-a)} \quad \ldots(3.22)$$

$$\|x'(t, \epsilon) - v'(t, \epsilon)\| \leqslant \epsilon M_5 e^{m(t-a)}. \quad \ldots(3.23)$$

Also from Theorem 3.2 it follows that

$$\|v(t, \epsilon) - w(t, \epsilon)\| \leqslant \epsilon M_6 \| B(\epsilon) - w(a, \epsilon)\| X [e^{-(t-a)} - e^{\delta_3 (t-a)/\epsilon}] \quad \ldots(3.24)$$

$$\|v'(t, \epsilon) - w'(t, \epsilon)\| \leqslant \epsilon M_7 \| B(\epsilon) - w'(a, \epsilon)\| e^{m(t-a)} + \|B(\epsilon) - w'(a, \epsilon)\| e^{-\delta_3 (t-a)/\epsilon}. \quad \ldots(3.25)$$

Finally we have

$$\|x(t, \epsilon) - u(t)\| \leqslant \|x(t, \epsilon) - w(t, \epsilon)\| + \|w(t, \epsilon) - u(t)\| \quad \ldots(3.26)$$

and

$$\|\mathbf{x}'(\mathbf{r}, \epsilon) - \mathbf{u}'(t)\| \leqslant \|\mathbf{x}'(t, \epsilon) - \mathbf{w}'(t, \epsilon)\| \qquad \ldots(3.27)$$
$$+ \|\mathbf{w}'(t, \epsilon) - \mathbf{u}'(t)\|.$$

The results (3.17)—(3.18) follow immediately from (3.22)—(3.27) and (3.15).

4. Estimates and Asymptotic Behaviour of Solutions of Linear Systems—II

In the previous section we obtained various estimates and in turn discussed the asymptotic behaviour of solutions of the linear system (2.19) under the assumption that the conditions (i)-(iii) of Theorem 2.4 hold good. The condition (i) of this theorem is a very important one and is imposed invariably for coupled systems of DEs. In fact, it is the quasimonotonicity condition for finite systems of DEs. Then if the condition (i) of Theorem 2.4 is not met by the system (2.19) one can still obtain the same limiting behaviour by adjoining a new system of DEs to (2.19) as follows. Following Müller [8, 9] we adjoin the following system to (2.19).

$$\begin{cases} -\operatorname*{Sup}_{H}\,[x' - \mu'] = O, \operatorname*{Inf}_{H}\,[\overline{x}' - \mu'] = O \\ -\operatorname{Sup}\,[\epsilon\, y' + \alpha\, y + \beta\, \mu] = -\gamma \\ \operatorname{Inf}\,[\epsilon\, \overline{y}' + \alpha\, \overline{y} + \beta\, \mu] = \gamma \\ -\mathrm{x}\,(a, \epsilon) = -\mathbf{A}\,(\epsilon),\ \overline{\mathrm{x}}\,(a, \epsilon) = \mathbf{A}\,(\epsilon) \\ -\mathbf{y}\,(a, \epsilon) = -\mathbf{B}\,(\epsilon),\ \overline{\mathbf{y}}\,(a, \epsilon) = \mathbf{B}\,(\epsilon) \end{cases} \qquad \ldots(4.1)$$

where

$$H := \{\mu, \mu' : x \leqslant \mu \leqslant \overline{\mathrm{x}}, y \leqslant \mu' \leqslant \overline{\mathrm{y}}\}.$$

Define

$$\beta^{+}_{ij} := \begin{cases} \beta_{ij} \text{ if } \beta_{ij} \geqslant 0 \\ 0 \text{ otherwise,} \end{cases}$$

$$\beta^{-}_{ij} := \beta_{ij} - \beta^{+}_{ij}\,.$$

Set

$$\hat{x}_{2l} = \overline{x}_l,\ \hat{x}_{2l-1} = -x_l,\ \hat{y}_{2l} = \overline{y}_l,\ \hat{y}_{2l+1} = -y_l.$$

Then the above system (4.1) may now be written as

$$
\begin{cases}
\hat{P}_1(\hat{x}, \hat{y}) := \hat{x}' - \hat{y} = O \\
\hat{Q}_{12i+1}(\hat{x}, \hat{y}) := \epsilon \hat{y}_{2i+1} + \alpha_{ii} \hat{y}_{2i+1} - \sum_{j=-\infty}^{\infty} \beta^{+}_{ij} \hat{x}_{2j} \\
\qquad + \sum_{j=-\infty}^{\infty} \beta^{-}_{ij} \hat{x}_{2j+1} = -\gamma_i \\
\hat{Q}_{12i}(\hat{x}\ \hat{y}) := \epsilon \hat{y}_{2i} + \alpha_{ii} \hat{y}_{2i} - \sum_{j=-\infty}^{\infty} \beta^{+}_{ij} x_{2j+1} \\
\qquad + \sum_{j=-\infty}^{\infty} \beta^{-}_{ij} \hat{x}_{2j} = \gamma_i \\
\hat{x}_{2i+1}(a, \epsilon) = -A_i(\epsilon),\ \check{x}_{2i}(a, \epsilon) = A_i(\epsilon) \\
\hat{y}_{2i+1}(a, \epsilon) = -B_i(\epsilon),\ \hat{y}_{2i}(a, \epsilon) = B_i(\epsilon)\ i \in Z.
\end{cases} \qquad \ldots(4.2)
$$

It can be easily seen that the the above system (4.2) satisfy a condition similar to that of (i) of Theorem 2.4. Also if (x, y) is a solution of the system (2.19) then $(\hat{x}, \hat{y})$ defined below is a solution of the new system (4.2):

$$\hat{x}_{2i+1} = -x_i,\ \hat{x}_{2i} = x_i,\ \hat{y}_{2i+1} = y_i,\ \hat{y}_{2i} = y_i,\ i \in Z.$$

The following theorem correspond to Theorem 3.3 in the unrestricted case discussed above. It is to be noted that the Theorem 3.3 was proved under the validity of the assumptions of Theorem 2.4.

Theorem 4.1—Assume that all the hypotheses of Theorem 3.3 except the hypothesis. (*i*) of theorem 2.4 are satisfied for the IVP (2.17)—(2.18). Then also the conclusions (3.19)—(3.20) for the IVP hold true.

PROOF —Consider the IVP (2.19). If this system does not satisfy the condition (i) of Theorem 2.4 then adjoin the system (4.2) to it. Following the procedure given in Theorem 3.3. We may arrive at the following estimates for the system (4.2)

$$
\begin{aligned}
| \hat{x}_i - \hat{u}_i | &\leqslant \epsilon M_7\, e^{m(t-a)} + \epsilon\, M_8 \| \hat{\mathbf{B}}(\epsilon) - \hat{\mathbf{u}}'(a) \| \\
&\qquad [e^{m(t-a)} - e^{-\delta_3(t-a)/\epsilon}] + D(\epsilon) \\
| \hat{y}_{2i} - \hat{u}_{2i} | &\leqslant \epsilon\, M_9\, e^{m(t-a)} + \epsilon\, M_{10} \| \hat{\mathbf{B}}(\epsilon) - \hat{\mathbf{u}}'(a) \|\, e^{m(t-a)} \\
&\qquad + \| \mathbf{B}(\epsilon) - \hat{\mathbf{u}}'(a) \|\, e^{-\delta_3(t-a)/\epsilon} + E(\epsilon) \\
| \hat{y}_{2i+1} - \hat{\mathbf{u}}_{2i+1} | &\leqslant \epsilon\, M_9\, e^{m(t-a)} + \epsilon\, M_{10} \| \hat{\mathbf{B}}(\epsilon) - \hat{\mathbf{u}}'(a) \|\, e^{m(t-a)} \\
&\qquad + \| \hat{\mathbf{B}}(\epsilon) - \hat{\mathbf{u}}'(a) \|\, e^{-\delta_3(t-a)/\epsilon} + F(\epsilon)\ i \in Z
\end{aligned}
$$

where $D(\epsilon)$, $E(\epsilon)$ $F(\epsilon)$ tend to zero as $\epsilon \to 0+$ and $\hat{\mathbf{u}}$ is the solution of the IVP

$$\begin{cases} \alpha_{ii}(t, 0)\, \check{u}_{2i+1} - \sum_{j=-\infty}^{\infty} \beta^{+}_{ij}(t, 0)\, \hat{u}_{2j} + \sum_{j=-\infty}^{\infty} \beta^{-}_{ij}(t, 0)\, \hat{u}_{2j+1} = -\gamma_i \\ \alpha_{ii}(t, 0)\, \hat{u}_{2i} - \sum_{j=-\infty}^{\infty} \beta^{+}_{ij}(t, 0)\, \hat{u}_{2j+1} + \sum_{j=-\infty}^{\infty} \beta^{-}_{ij}(t, 0)\, \hat{u}_{2j} = \gamma_i \\ \hat{\mathbf{u}}(a) = \hat{\mathbf{A}}(0)_i,\ i \in Z. \end{cases} \qquad \ldots(4.3)$$

It can be verified that if $(\mathbf{x}, \mathbf{y})$ and $\mathbf{u}$ are respectively solutions of the system (2.9) and (3.16) then $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ and $\hat{\mathbf{u}}$ defined by

$$\hat{x}_{2i} = x_i,\ \hat{x}_{2i+1} = -x_i,\ \hat{y}_{2i} = y_i,\ \hat{y}_{2i+1} = -y_i$$

$$\hat{u}_{2i} = u_i,\ \hat{u}_{2i+1} = -u_i,\ i \in Z$$

are respectively solutions of the IVPs (4.2) and (4.3). These observations complete the proof of this theorem.

5. Estimates and Asymptotic Behaviour of Solutions of Nonlinear Systems

In Section 4 different estimates are obtained for solutions and their derivatives for the linear system (2.17)—(2.18). Guided by the experience with the linear system we now proceed to consider the nonlinear system (1.3) and derive estimates for solutions of the same. The estimates thus derived may contain boundary layer terms which explicitly describe the nature of the nonuniform behaviour of solutions of the nonlinear system (1.3) in t and ϵ.

The reduced problem for the IVP is given by

$$\begin{cases} \mathbf{p}' - \mathbf{u}(t, \mathbf{p}, \mathbf{q}\ 0) = \mathbf{O},\ t \in J_0 \\ \mathbf{v}(t, \mathbf{p}, \mathbf{q}, 0) = \mathbf{O} \\ \mathbf{p}(a) = \mathbf{A}(0). \end{cases} \qquad \ldots(5.1)$$

We shall now make the following assumptions.

(ii The functions $\mathbf{u}$, $\mathbf{v}$ of (1.3) satisfy the condition (i) of Theorem 2.3.

(ii) There exists a function $\mathbf{Y}$ such that

$$\mathbf{v}(t, \mathbf{p}, \mathbf{Y}, O) = \mathbf{O} \qquad \ldots(5.2)$$

and the resulting IVP

$$\begin{cases} \mathbf{p}' - \mathbf{u}(t, \mathbf{p}, \mathbf{Y}, 0) = \mathbf{O} \\ \mathbf{p}(a) = \mathbf{A}(0), \end{cases} \qquad \ldots(5.3)$$

has a unique solution X such that

$$\|\mathbf{Y}'\| \leqslant M_9 \qquad \text{.. (5.4)}$$

$$\begin{cases} \mathbf{v}(t, \mathbf{X}+\alpha, \mathbf{Y}+\beta, \epsilon) - \mathbf{v}(t, \mathbf{X}+\alpha, \mathbf{Y}, \epsilon) \leqslant -K\beta \\ \mathbf{v}(t, \mathbf{X}-\alpha, \mathbf{Y}-\beta, \epsilon) - \mathbf{v}(t, \mathbf{X}-\alpha, \mathbf{Y}, \epsilon) \geqslant K\beta \end{cases} \qquad \text{...(5.5)}$$

where $k > 0$, for every nonnegative functions α, β.

(iii) For some constant $1 > 0$ and for every nonnegative functions α, α^*, β, β^*, $\alpha \leqslant \alpha^*$, $\beta \leqslant \beta^*$

$$\begin{cases} \mathbf{u}(t, \mathbf{X}+\alpha^*, \mathbf{Y}+\beta^*, \epsilon) - \mathbf{u}(t, \mathbf{X}+\alpha, \mathbf{Y}+\beta, \epsilon) \\ \qquad \leqslant 1(\alpha^* - \alpha + \beta^* - \beta) \\ \mathbf{u}(t, \mathbf{X}-\alpha, \mathbf{Y}-\beta, \epsilon) - \mathbf{u}(t, \mathbf{X}-\alpha^*, \mathbf{Y}-\beta^*, \epsilon) \\ \qquad \leqslant 1(\alpha^* - \alpha + \beta^* - \beta) \end{cases} \qquad \text{...(5.6)}$$

$$\begin{cases} \mathbf{v}(t, \mathbf{X}+\alpha^*, \mathbf{Y}+\beta^*, \epsilon) - \mathbf{v}(t, \mathbf{X}+\alpha, \mathbf{Y}+\beta, \epsilon) \\ \qquad \leqslant 1(\alpha^* - \alpha + \beta^* - \beta) \\ \mathbf{v}(t, \mathbf{X}-\alpha, \mathbf{Y}-\beta, \epsilon) - \mathbf{v}(t, \mathbf{X}-\alpha^*, \mathbf{Y}-\beta^*, \epsilon) \\ \qquad \leqslant 1(\alpha^* - \alpha + \beta^* - \beta). \end{cases} \qquad \text{...(5.7)}$$

Theorem 5.1—Let (x, y) and (X, Y) be respectively solutions of the IVPs (1.3) and (5.1). Further assume that

$$\|\mathbf{u}(t, \mathbf{X}, \mathbf{Y}, \epsilon) - \mathbf{u}(t, \mathbf{X}, \mathbf{Y}, 0)\| \leqslant \epsilon M_7 \qquad \text{...(5.8)}$$

and

$$\|\mathbf{v}(t, \mathbf{X}, \mathbf{Y}, \epsilon)\| \leqslant \epsilon M_8. \qquad \text{...(5.9)}$$

Then under the above assumptions

(i) — (ii) we have

$$\begin{aligned} \|\mathbf{x}(t, \epsilon) - \mathbf{X}(t)\| &\leqslant \epsilon M_3 e^{m(t-a)} \\ &+ \epsilon M_4 \|\mathbf{B}(\epsilon)\mathbf{Y}(a)\| \\ &\times [e^{m(t-a)} - e^{-k(t-a)/_8}] \end{aligned} \qquad \text{...(5.10)}$$

$$\begin{aligned} \|\mathbf{y}(t, \epsilon) - \mathbf{Y}(t)\| &\leqslant \epsilon M_5 e^{m(t-a)} \\ &+ \epsilon M_6 \|\mathbf{B}(\epsilon) - \mathbf{Y}(a)\| \times e^{m(t-a)} \\ &+ \|\mathbf{B}(\epsilon) - \mathbf{Y}(a)\| e^{-k(t-a)/_\epsilon}, m > 0. \end{aligned} \qquad \text{...(5.11)}$$

PROOF : Because of the conditions (i)—(iii) stated above, Theorem 2.3 is applicable to the IVP (1.3). Also we have

$$\|\mathbf{A}(\epsilon) - \mathbf{A}(0)\| \leqslant \epsilon M_3.$$

Consider the vector functions (x, y) and $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ defined by

$$\bar{x} = X + \eta, \ \bar{y} = Y + \gamma$$

$$x = X - \eta, \ y = Y - \gamma$$

where

$$\eta = (\ldots, \eta, \eta, \ldots), \ \gamma = (\ldots, \gamma, \gamma, \ldots)$$

$$\eta(t, \epsilon) = \epsilon M_3 e^{m(t-a)} + \epsilon M_4 \|B(\epsilon) - Y(a)\| X \\ [e^{m(t-a)} - e^{-k(t-a)/\epsilon}] \quad \ldots(5.13)$$

$$\gamma(t, \epsilon) = \epsilon M_5 e^{m(t-a)} + \epsilon M_6 \| B(\epsilon) - Y(a)\| e^{m(t-a)} \\ + \|B(\epsilon) - Y(a)\| e^{-k(t-a)/\epsilon}. \quad \ldots(5.14)$$

We have

$$P_i'(\bar{x}, \bar{y}) := X_i'(t) + \eta' - [u_i(t, X + \eta, Y + \gamma, \epsilon) \\ - u_i(t, X, Y, \epsilon)] - u_i(t, X, Y, \epsilon) \\ \geqslant -\epsilon M_7 + \eta' - l(\eta + \gamma),$$

from (5.8), (5.3) and (5.6)

$$\geqslant \epsilon e^{m(t-a)} [-M_7 + m M_3 - l M_3 - l M_5] \\ + \epsilon \| B(\epsilon) - Y(a)\| e^{m(t-a)} [m M_4 - l M_4 - l M_6] \\ + \|B(\epsilon) - Y(a)\| e^{-k(t-a)} [k M_4 + l \epsilon M_4 - 1]$$

by (5.13)–(5.14), $i \in Z$.

Choose M_4, M_5, M_6 and m such that

$$k M_4 - 1 > 0, \quad \ldots(5.15)$$

$$k M_5 - M_8 - M_9 - l M_3 > 0 \quad \ldots(5.16)$$

$$-l M_4 + k M_6 > 0 \quad \ldots(5.17)$$

and

$$\left.\begin{array}{l} m M_3 - M_7 - l M_3 - l M_5 > 0 \\ m M_4 - l M_4 - l M_6 > 0. \end{array}\right\} \quad \ldots(5.18)$$

Therefore we have,

$$P_i'(\bar{x}, \bar{y}) \geqslant 0 = P_i'(x, y). \ i \in Z$$

that is,

$$P'(\bar{x}, \bar{y}) \geqslant O = P(x, y).$$

Also

$$Q'_i\ (\bar{x}, \bar{y}) = \epsilon Y'_i\ (t) + \epsilon\ \gamma'\ (t, \epsilon)$$

$$- [v_i\ (t, X + \eta, Y + \gamma, \epsilon) - v_i\ (t, X, Y + \gamma, \epsilon)$$

$$+ v_i\ (t, X, Y + \gamma, \epsilon) - v_i\ (t, X, Y, \epsilon)]$$

$$- v_i\ (t, X, Y, \epsilon)$$

$$\geqslant - \epsilon\ M_9 - \epsilon\ M_8 + \epsilon\ \gamma' - 1\eta + k\ \gamma$$

by (5.4), (5.9), (5.5) and (5.7)

$$\geqslant \epsilon\ e^{m(t-a)} [- M_9 - M_8 + \epsilon m\ M_5 - 1\ M_3 + k\ M_5]$$

$$+ \epsilon\ \|B\ (\epsilon) - Y\ (a)\|\ e^{m(t-a)}\ [\epsilon\ m\ M_6 - 1\ M_4 - k\ M_6]$$

$$+ \|\ B\ (\epsilon) - Y\ (a)\ \|\ e^{-k(t-a)/\epsilon}\ [- k + 1\ \epsilon\ M_4 + k]$$

by (5.13) and (5.14) $i \in Z$.

Using (5.15) – (5.18) we have

$$Q'_i\ (\bar{x}, \bar{y}) \geqslant 0 = Q_i\ (x, y),\ i \in Z.$$

Hence

$$Q'\ (\bar{x}, \bar{y} \geqslant Q'\ (x, y).$$

Again

$$\bar{x}_i\ (a, \epsilon) = X_i\ (a) + \epsilon\ M_3 = A_i\ (0) + \epsilon\ M_3 \geqslant A_i\ (\epsilon) = x_i\ (a, \epsilon),\ i \in Z.$$

Similarly,

$$\bar{y}_i\ (a, \epsilon) \geqslant y_i\ (a, \epsilon),\ i \in Z.$$

That is we established the following inequalities

$$\left.\begin{array}{l} P'\ (x, y) \leqslant P\ (\bar{x}, \bar{y}) \\ Q\ (x, y) \leqslant Q'\ (\bar{x}, \bar{y}) \\ R'\ (x, y) \leqslant R'\ (\bar{x}, \bar{y}) \end{array}\right\} \qquad \ldots(5.19)$$

which, by Theorem 2.3, yield

$$x\ (t, \epsilon) \leqslant \bar{x}\ (t, \epsilon),\ y\ (t, \epsilon) \leqslant \bar{y}\ (t, \epsilon),\ t \in J. \qquad \ldots(5.20)$$

Similar steps yield

$$x\ (t, \epsilon) \leqslant x\ (t, \epsilon)\ y\ (t, \epsilon) \leqslant y\ (t, \epsilon),\ t \in J.$$

The inequalities (5.10) – (5.11) follow from (5.20) – (5.21). Thus we have

$$\lim_{\epsilon \to 0+} \mathbf{x}(t, \epsilon) = \mathbf{X}(t), \ t \in J$$

$$\lim_{\epsilon \to 0+} \mathbf{y}(t, \epsilon) = \mathbf{Y}(t), \ t \in J_0.$$

Conclusions

The analysis of the previous section can be carried over to study IVPs for higher order equations of the form

$$\epsilon \, \mathbf{y}^{(n)} = \mathbf{f}(t, \mathbf{y}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n-1)}), \ t \in J_0$$

subject to ICs

$$\mathbf{y}(a, \epsilon) = \mathbf{A}^1(\epsilon), \ldots, \mathbf{y}^{(n-1)}(a, \epsilon) = A^{(n)}(\epsilon),$$

where $\epsilon > 0$ is a small parameter,

$$y^{(n)} := d^n y/dt^n, \ \mathbf{A}^i(\epsilon) = A^i(0) + 0(\epsilon), \ i \in Z.$$

The method adopted in discussing non-quasimonotone linear system can be used to discuss nonquasimonotone nonlinear systems

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SUMS INVOLVING THE LARGEST PRIME DIVISOR OF AN INTEGER II

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For $n \geqslant 2$, let $P(n)$ denote the largest prime divisor of an integer. In this paper, we develop an elementary method for estimating the sum $\sum_{2 \leqslant n \leqslant x} f(n) P(n)$ where $f(n)$ is a multiplicative arithmetical function.

1. INTRODUCTION

Let $P(n)$ denote the largest prime divisor of an integer $n \geqslant 2$. Various sums involving $P(n)$ have been studied by many, authors[1-5,7,8,10-12,15]. In this paper, we discuss sums of the form $\sum_{2 < n \leqslant x} f(n) P(n)$ where $f(n)$ is a multiplicative function. Our methods are elementary and can also be used to estimate sums of the form $\sum_{2 \leqslant n \leqslant x} f(n) g(n)$ where $f(n)$ is multiplicative and $g(n)$ is additive. Through analytic methods are available for estimating the latter sums[9,4,13,14], it appears that elementary methods are best suited for the sums discussed in this paper.

2. PRELIMINARIES

Let $\mu(n)$ denote the Möbius function $\phi(n)$ be the Euler totient function, $\sigma(n)$ be the sum of all the positive divisors of n, $\omega(n)$ be the number of distinct prime factors of n and for $k \geqslant 1$, $d_k(n)$ be the Piltz divisor function defined to be the number of ordered k-tuples $(x_1, x_2, \ldots, x_l)$ of positive integers such that $x_1, x_2, x_k = n$. Also let $\beta(n) = \sum_{p \mid n} p$ and $B(n) = \sum_{p^\alpha \| n} \alpha p$ where, as usual, P denotes a prime number and $p^\alpha \| n$ means that $p^\alpha | n$ and $p^{\alpha+1} \nmid n$. Further for any function g defined at primes, let the function G be defined by $G(I) = 0$ and for $n > 1$

$$G(n) = \sum_{p \mid n} g(p). \qquad \ldots(2.1)$$

In the sequel, $\sum'_{n \leqslant x}$ means $\sum_{2 < n \leqslant x}$.

Lemma 2.1—As $x \to \infty$, we have

$$\sum_{n\leqslant x}{}' P(n) = \sum_{n<x} \beta(n) + O\left[\frac{x^{3/2}}{\log x}\right] \qquad \ldots(2.2)$$

and

$$\sum_{n\leqslant x} B(n) = \sum_{n<x} \beta(n) + O(x \log\log x). \qquad \ldots(2.3)$$

Proof : We have

$$\sum_{n\leqslant x} P(n) = \sum_{pm\leqslant x, p(m)<p} p = \sum_{\substack{pm\leqslant x \\ p(m)<p, p\leqslant\sqrt{x}}} P + \sum_{\substack{pm<x \\ p(m)\leqslant p, p)>\sqrt{x}}} p$$

$$= O\left[\sum_{pm\leqslant x, p\leqslant\sqrt{x}} p\right] + \sum_{pm<x} p - \sum_{\substack{pm<x \\ p<\sqrt{x}}} p$$

$$= \sum_{n\leqslant x} \beta(n) + O\left[\sum_{p<\sqrt{x}} p \frac{x}{p}\right]$$

and (2.2) follows. On noting that

$$\sum_{n\leqslant x} B(n) = \sum_{n<x} \beta(n) + \sum_{p^{\alpha} m\leqslant x, \alpha\geqslant 2} \alpha p$$

$$= \sum_{n\leqslant x} \beta(n) + O\left[x \sum_{p^{\alpha}\leqslant x, \alpha\geqslant 2} \alpha/p^{\alpha-1}\right]$$

we get (2.3).

Theorem 2.1—Let $f(n)$ be an arithmetical function such that $f(n) \ll g(n)$ where $g(x)$ is positive valued and increasing on $[1, \infty]$. Then

$$\sum_{n\leqslant x} f(n)\beta(n) = \sum_{n\leqslant x}{}' f(n) P(n) + O(g(x) x^{3/2}/\log x). \qquad \ldots(2.4)$$

$$\sum_{n\leqslant x} f(n) B(n) = \sum_{n\leqslant x} f(n)\beta(n) + O(g(x) x \log\log x) \qquad \ldots(2.5)$$

$$\sum_{n\leqslant x} f(n) B(n) = \sum_{n\leqslant x}{}' f(n) P(n) + O(g(x) x^{3/2}/\log x). \qquad \ldots(2.6)$$

Proof : Since

$$\sum_{n\leqslant x} f(n)\beta(n) = \sum_{n\leqslant x}{}' f(n) P(n) + \sum_{n\leqslant x}{}' f(n)(\beta(n) - P(n)) + O(1)$$

$$= \sum_{n<x}{}' f(n) P(n) + O\Big(g(x) \sum_{n\leqslant x}{}' (\beta(n) - P(n))\Big) + O(1)$$

(2.4) follows from (2.2). Similarly (2.5) follows from (2.3) and (2.6) follows from (2.4) and (2.5).

Remark 2.1 : Theorem 2.1 extends and sharpens an earlier result due to (De Koninck and Ivić[5], Lemma 3).

Lemma 2.2—Let $G(n)$ be as defined in (2.1). Then

$$\sum_{n\leqslant x} d_k(n)\, G(n) = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \Big[\sum_{p^i n\leqslant x} g(P)\, d_k(n)\Big].$$

PROOF : We recall (cf. Ivic and Pomerance[15], p. 4-5) that $d_k(n)$ is multiplicative and $\hat{d}_k(p^\alpha) = \left[\begin{matrix}\alpha+k-1\\ \alpha\end{matrix}\right]$ for prime p and positive integral α. Also we note the following identity :

$$\sum_{i,j\geqslant 0,\, i+j=n} (-1)\, i \left[\begin{matrix}k\\ i\end{matrix}\right] \left[\begin{matrix}k+j-1\\ j\end{matrix}\right] = 0 \text{ for } k \geqslant 1 \text{ and } n \geqslant 1. \quad \text{...(2.7)}$$

This follows from

$$1 = (1-z)^k (1-z)^{-k} = \left\{\sum_{i=0}^{k} (-1)^i \left[\begin{matrix}k\\ i\end{matrix}\right] z^i\right\} \left\{\sum_{j=0}^{k} \left[\begin{matrix}k+j-1\\ j\end{matrix}\right] z^j\right\}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{i,j\geqslant 0, i+j=n} (-1)^i \left[\begin{matrix}k\\ i\end{matrix}\right]\left[\begin{matrix}k+j-1\\ j\end{matrix}\right]\right\} z^n.$$

Now by (2.1)

$$\sum_{n\leqslant x} d_k(n)\, G(n) = \sum_{n\leqslant x} d_k(n) \sum_{p\mid n} g(p) = \sum_{pn\leqslant x} g(p)\, d_k(pn)$$

$$= \sum_{pn\leqslant x, p\nmid n} g(p)\, d_k(n) + \sum_{pn\leqslant x, p\mid n} g(p)\, d_k(pn)$$

$$= \left[\begin{matrix}k\\ i\end{matrix}\right] \left\{ \sum_{pn\leqslant x} g(p)\, d_k(n) - \sum_{pn\leqslant x, p\mid n} g(p)\, d_k(n) \right\}$$

$$+ \sum_{pn\leqslant x, p\mid n} g(p)\, d_k(pn)$$

$$= \left[\begin{matrix}k\\ 1\end{matrix}\right] \sum_{pn\leqslant x} g(p)\, d_k(n) + \sum_{p^2 n\leqslant x} g(p) \Big\{ d_k(P^2 n)$$

$$- \left[\begin{matrix}k\\ 1\end{matrix}\right] d_k(pn) \Big\}$$

$$= \left[\begin{matrix}k\\ 1\end{matrix}\right] \sum_{pn\leqslant x} g(p)\, d_k(n) + \Sigma_1 \quad \text{...(2.8)}$$

say, Now since $\left[\begin{matrix} k+1 \\ 2 \end{matrix}\right] - \left[\begin{matrix} k \\ 1 \end{matrix}\right]^2 = -\left[\begin{matrix} k \\ 2 \end{matrix}\right]$, we have

$$\sum\nolimits_1 = -\left[\begin{matrix} k \\ 2 \end{matrix}\right] \sum_{p^2 n < x, p \nmid n} g(p)\, d_k(n) + \sum_{p^3 n \leqslant x} g(p) \Big\{ d_k(p^3 n)$$

$$= -\left[\begin{matrix} k \\ 1 \end{matrix}\right] d_k(p^n\, n) = -\left\{\begin{matrix} k \\ 2 \end{matrix}\right\} \sum_{p^2 n < x} g(p)\, d_k(n) + \sum_{p^3 n < x} g(p)$$

$$\Big\{ d_k(p^3 n) - \left[\begin{matrix} k \\ 1 \end{matrix}\right] d_k(p^2 n) + \left[\begin{matrix} k \\ 2 \end{matrix}\right] d_k(pn) \Big\}.$$

Continuing this, we arrive, by (2.8), at

$$\sum_{n \leqslant x} d_k(n)\, G(n) = \sum_{i=1}^{k} (-i)^{i-1} \left[\begin{matrix} k \\ i \end{matrix}\right] \sum_{p^i n < x} g(p)\, d_k(n)$$

$$+ \sum_{p^{k+1} n \leqslant x} g(p) \Big\{ d_k(p^{k+1}) - \left[\begin{matrix} k \\ 1 \end{matrix}\right] d_k(p^k n) + \ldots + (-1)^k \left[\begin{matrix} k \\ k \end{matrix}\right] d_k(pn) \Big\}$$

However, this last sum is zero since for any prime p and any integer $\alpha \geqslant 0$

$$d_k(p^{k+1+\alpha}) - \left[\begin{matrix} k \\ 1 \end{matrix}\right] d_k(p^{k+\alpha}) + \ldots + (-1)^k \left[\begin{matrix} k \\ k \end{matrix}\right] d_k(p^{1+\alpha}) = 0.$$

In fact, this is equivalent to

$$\left[\begin{matrix} \overline{k+1+\alpha}+k-1 \\ k+1+\alpha \end{matrix}\right] - \binom{k}{1} \left[\begin{matrix} k+\alpha+k-1 \\ k+\alpha \end{matrix}\right] + \ldots + (-1)\, k - \left[\begin{matrix} k \\ k \end{matrix}\right]$$

$$\left[\begin{matrix} \overline{1+\alpha}+k-1 \\ 1+\alpha \end{matrix}\right] = 0$$

which is the case $n = k+1+\alpha$ of the identity (2.7). This completes the proof of the Lemma.

Lemma 2.3—Let G be defined as in (2.1). Then

$$\sum_{n < x} \mu(n)\, G(n) = -\sum_{p^\alpha n \leqslant x} g(p)\, \mu(n) \qquad \ldots(2.9)$$

$$\sum_{n > x} \phi(n)\, G(n) = \sum_{p^\alpha n \leqslant x} (p-1)\, g(p)\, \phi(n) \qquad \ldots(2.10)$$

and if F is any multiplicative function satisfying

$$F(p^{\alpha+2}) + pF(p^\alpha) = F(p)\, F(p^{\alpha+1}) \qquad \ldots(2.11)$$

for all primes and positive integers α, then

$$\sum_{n<x} F(n)\, G(n) = \sum_{pn \leqslant x} g(p)\, F(p)\, F(n) - \sum_{p^2 n \leqslant x} pg(p)\, F(n). \quad \ldots(2.12)$$

PROOF : We prove (2.9), the proofs of (2.10) and (2.12) being similar.

$$\sum_{n<x} \mu(n)\, G(n) = \sum_{n \leqslant x} \mu(n) \sum_{p \mid n} g(p) = \sum_{pn \leqslant x} g(P)\, \mu(pn)$$

$$= - \sum_{pn \leqslant x, p \nmid n} g(p)\, \mu(n) = - \sum_{pn \leqslant x} g(p)\, \mu(n)$$

$$+ \sum_{pn \leqslant x, p \mid n} g(p)\, \mu(n)$$

$$= - \sum_{pn < x} g(p)\, \mu(n) + \sum_{p^2 n < x} g(p)\, \mu(pn)$$

$$= - \sum_{pn < x} g(p)\, \mu(n) - \sum_{p^2 n \leqslant x, p \nmid n} g(p)\, \mu(n)$$

$$= - \sum_{pn \leqslant x} g(p)\, \mu(n) - \sum_{p^2 n \leqslant x} g(p)\, \mu(n)$$

$$+ \sum_{p^2 n \leqslant x} g(p)\, \mu(n).$$

Continuing this, we get (2.9).

3. MAIN RESULTS

In the following, N denotes an arbitrary but fixed positive integer.

Theorem 3.1—There exist constants $a_0 = \dfrac{\pi^2}{12}$, $a_1, \ldots, a_{N-1}$ such that

$$\sum_{n \leqslant x}{}' P(n) = x^2 \sum_{i=1}^{N} \frac{a_{i-1}}{(\log x)^i} + O(x^2/\log x)^{N+1}) \quad \ldots(3.1)$$

$$\sum_{n<x} \beta(n) = x^2 \sum_{i=1}^{N} \frac{a_{i-1}}{(\log x)^i} + O(x^2/(\log x)^{N+1}) \quad \ldots(3.2)$$

and

$$\sum_{n \leqslant x} B(n) = x^2 \sum_{i=1}^{N} \frac{a_{i-1}}{(\log x)^i} + O(x^2/(\log x)^{N+1}). \quad \ldots(3.3)$$

Further, we have

$$a_i = \int_1^{\infty} \frac{[t]\,(\log t)^i}{t^3}\, dt \quad \ldots(3.4)$$

where $[x]$ denotes the largest integer $\leqslant x$ and

$$a_0 < a_1 < a_2 < \ldots < a_{N-1}. \qquad \ldots(3.5)$$

PROOF : We prove (3.2) and note that Lemma 2.1 and (3.2) yield (3.1) and (3.3). By the prime number theorem and partial summation, we have

$$\sum_{p\leqslant x} p = \int_2^x \frac{t}{\log t}\,dt + O\left[\frac{x^2}{(\log x)^{N+1}}\right]. \qquad \ldots(3.6)$$

Hence by partial summation

$$\sum_{n\leqslant x} \beta(n) = \sum_{pn\leqslant x} p + \sum_{n\leqslant x/2} \Big[\sum_{p\leqslant x/n} p\Big]$$

$$= \sum_{n\leqslant x/2}\left\{\int_2^{x/n} \frac{t}{\log t}\,dt + O\left[\frac{x^2}{n^2(\log x/n)^{N+1}}\right]\right\}$$

$$= x^2\int_1^{x/2} \frac{[t]}{t^3\log(x/t)}\,dt + O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

$$= \frac{x^2}{\log x}\int_1^{x/2} \frac{[t]}{t^3}\left\{\sum_{i=0}^{N-1}\left[\frac{\log t}{\log x}\right]^i + O\left[\left[\frac{\log t}{\log i}\right]^N\right]\right\}dt + O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

$$= \frac{x^2}{\log x}\sum_{i=0}^{N-1}\frac{1}{(\log x)^i}\left\{\int_1^{\infty}\frac{[t](\log t)^i}{t^3}\,dt + O\left[\frac{(\log x)^i}{x}\right]\right\}$$

$$+ O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

$$= x^2\sum_{i=1}^{N}\frac{a_i - 1}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right]$$

where a_i is as given in (3.4).

To prove (3.5) we have by partial summation

$$\sum_{n\leqslant x}\frac{1}{n^2} = \frac{[x]}{x^2} + 2\int_1^x \frac{[t]}{t^3}\,dt.$$

So that on letting $x \to \infty$, we get $\zeta(2) = 2a_0$. By a similar argument, we find

$$\sum_{n=1}^{\infty}\frac{(\log n)^i}{n^2} = 2a_i - ia_{i-1},\ i \geqslant 1. \qquad \ldots(3.7)$$

It is easy to show that $\zeta(2) < 2 \sum_{n=1}^{\infty} \frac{\log n}{n^2}$ so that $a_0 < a_1$. For $i \geqslant 2$, we have by (3.7)

$$2a_i = ia_{i-1} + \sum_{n=2}^{\infty} \frac{(\log n)^i}{n^2} > ia_{i-1} \geqslant 2a_{i-1}$$

since $a_i \geqslant 0$ for all i. This proves (3.5) and thus the theorem.

Remark 3.1: Theorem 3 1 is due to (De Koninck and Ivić[5], Theorem 1) who sharpened an earlier result of Alladi and Erdös[2]. De Koninck and Ivić proved (3.1) and deduced (3.2) and (3.3). Our method of proof has an advantage in that it yields the representation (3.4) (and the inequalities given in (3.5)).

Theorem 3.2—There exist constants $d_0, d_1, \ldots, d_{N-1}$ such that

$$\sum_{n \leqslant x}{}' d_k(n) P(n) = x^2 \sum_{i=1}^{N} \frac{d_{i-1}}{(\log x)^i} + O\frac{x^2}{(\log x)^{N+1}}\Big] \qquad \ldots(3.8)$$

$$\sum_{n \leqslant x} d_k(n) B(n) = x^2 \sum_{i=1}^{N} \frac{d_{-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.9)$$

$$\sum_{n \leqslant x} d_k(n) B(n) = x^2 \sum_{i=1}^{N} \frac{d_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.10)$$

PROOF : First we prove (3.9). By Lemma 2.2, we have

$$\sum_{n \leqslant x} d_k(n) \beta(n) = \sum_{i=1}^{k} (-1)^{i-1} \left[\begin{matrix} k \\ i \end{matrix}\right] \sum_{p^i n \leqslant x} p d_k(n)$$

$$= \left[\begin{matrix} k \\ 1 \end{matrix}\right] \sum_{pn \leqslant x} p d_k(n) + \sum_{i=2}^{k} (-1)^{i-1} \left[\begin{matrix} k \\ i \end{matrix}\right]$$

$$\times \sum_{p^i n \leqslant x} p d_k(n)$$

$$= k\,\Sigma_1 + \sum_{i=2}^{k} (-1)^{i-1} \left[\begin{matrix} k \\ i \end{matrix}\right] \Sigma_i \qquad \ldots(3.11)$$

say. Since $\sum_{n\leqslant x} d_k(n) \ll_k x(\log x)^{k-1}$ (cf. Ivić and Pomerance[15], p. 263), we have for $i \geqslant 2$

$$\Sigma_i = \sum_{p^i n\leqslant x} p\, d_k(n) = \sum_{p^i\leqslant x} p \sum_{n\leqslant x/p^i} d_k(n)$$

$$\ll \sum_{p^i\leqslant x} p\left(\frac{x}{p^i}\right)\left(\log \frac{x}{p^i}\right)^{k-1} \ll x(\log x)^{k-1}\log\log x. \tag{3.12}$$

For the estimation of Σ_1, we note that by (3.6), for each positive integer r, there exist constants $b_1, b_2, \ldots, b_r$ such that

$$\sum_{p\leqslant x} p = x^2 \sum_{i=1}^{r} \frac{b_i}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{r+1}}\right]. \qquad \ldots(3.13)$$

Hence

$$\Sigma_1 = \sum_{pn\leqslant x, n\leqslant\sqrt{x}} p d_k(n) + \sum_{pn\leqslant x, n>\sqrt{x}} p d_k(n)$$

$$= \sum_{n\leqslant\sqrt{x}} d_k(n) \sum_{p\leqslant x/n} p + O\left[\sum_{n>\sqrt{x}} d_k(n)\,\frac{x^2}{n^2}\right]$$

$$= \sum_{n\leqslant\sqrt{x}} d_k(n)\left\{\frac{x^2}{n^2}\sum_{i=1}^{r} b_i\left[\log\frac{x}{n}\right]^{-i} + O\left[\frac{(x/n)^2}{(\log x/n)^{r+1}}\right]\right\}$$

$$+ O\left(x^{3/2}(\log x)^{k-1}\right)$$

$$= x^2\sum_{i=1}^{r}\frac{b_i}{(\log x)^i}\left\{\sum_{n\leqslant\sqrt{x}}\frac{d_k(n)}{n^2}\left[1-\frac{\log n}{\log x}\right]^{-i}\right\} + O\left[\frac{x^2}{(\log x)^{r+1}}\right]$$

$$= x^2\sum_{i=1}^{r}\frac{b_i}{(\log x)^i}\left\{\sum_{n\leqslant\sqrt{x}}\frac{d_k(n)}{n^2}\left[\sum_{j=0}^{r} b_{ij}\left[\frac{\log n}{\log x}\right]^j\right.\right.$$

$$\left.\left. + O\left[\left[\frac{\log n}{\log x}\right]^{r+1}\right]\right]\right\} + O\left[\frac{x^2}{(\log x)^{r+1}}\right]$$

$$= x^2\sum_{\substack{1\leqslant i\leqslant r\\ 0\leqslant j\leqslant r}}\frac{b_i b_{ij}}{(\log x)^{i+j}}\left\{\sum_{n=1}^{\infty}\frac{d_k(n)(\log n)^j}{n^2} + O\left[\frac{(\log x)^{k-1}}{x^{1/2}}\right]\right\}$$

$$+ O\left[\frac{x^2}{(\log x)^{r+1}}\right]$$

$$= x^2\sum_{i=1}^{r}\frac{c_i}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{r+1}}\right].$$

Now (3.9) follows from (3.11), (3.12) and the above.

On taking $f(n) = d_k(n)$, $g(x) = x^\epsilon$ and recalling that $d_k(n) \ll_\epsilon n^\epsilon$ for each $\epsilon > 0$) in Theorem 2.1, we obtain (3.8) and (3.10) from (3.9).

Theorem 3.3—There exist constants e_i, f_i, g_i, $0 \leqslant i \leqslant N-1$ such that

$$\sum_{n \leqslant x}{}' \mu(n) P(n) = x^2 \sum_{i=1}^{N} \frac{e_{i-1}}{\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.14)$$

$$\sum_{n \leqslant x} \mu(n) \beta(n) = x^2 \sum_{i=1}^{N} \frac{e_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.15)$$

$$\sum_{n \leqslant x} \mu(n) B(n) = x^2 \sum_{i=1}^{N} \frac{e_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.16)$$

$$\sum_{n \leqslant x}{}' \phi(n) P(n) = x^3 \sum_{i=1}^{N} \frac{f_{i-1}}{(\log x)^i} + O\left[\frac{x^2}{(\log x)^{N+1}}\right] \qquad \ldots(3.17)$$

$$\sum_{n \leqslant x} \phi(n) \beta(n) = x^3 \sum_{i=1}^{N} \frac{f_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.18)$$

$$\sum_{n \leqslant x} \mu(n) B(n) = x^3 \sum_{i=1}^{N} \frac{f_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.19)$$

$$\sum_{n \leqslant x}{}' \sigma(n) P(n) = x^3 \sum_{i=1}^{N} \frac{g_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.20)$$

$$\sum_{n \leqslant x} \mu(n) \beta(n) = x^3 \sum_{i=1}^{N} \frac{g_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right] \qquad \ldots(3.21)$$

and

$$\sum_{n \leqslant x} \sigma(n) B(n) = x^3 \sum_{i=1}^{N} \frac{g_{i-1}}{(\log x)^i} + O\left[\frac{x^3}{(\log x)^{N+1}}\right]. \qquad (3.22)$$

PROOF : The proofs of (3.15), (3.18) and (3.21) are similar to that of (3.9). We use (2.9), (2.10) and (2.12) in turn, instead of Lemma 2.2. The remaining assertions of the theorem follow from Theorem 2.1.

4. Other Results

Our general summation formulae given in Lemmas 2.2 and 2.3 and our methods of Section 2 can be used to discuss sums of the form $\sum\limits_{n \leqslant x} f(n)\, g(n)$ where $f(n)$ is either d_k or $\mu(n)$ or $\phi(n)$ or $\sigma(n)$ and $g(n)$ is an additive function. As a further example, we have

Theorem 4.1—For any integers $k \geqslant 2$ and $N \geqslant 1$, there exist constants A_i, B_i and C_j, $0 \leqslant i \leqslant k-1$ and $0 \leqslant j \leqslant N-1$ such that

$$\sum_{n \leqslant x} d_k(n)\, \omega(n) = x \log\log x \sum_{i=0}^{k-1} A_i (\log x)^{k-1-i} + x \sum_{i=0}^{k-1} B_i (\log x)^{k-1-i}$$

$$+ x \sum_{i=0}^{N-1} \frac{C_i}{(\log x)^i} + O\left[\frac{x}{(\log x)^N}\right].$$

Proof : It is well known (cf. Ivić and Pomerance[15], p. 163) that for certain constants $a_i^{(k)}$, $0 \leqslant i \leqslant k-1$

$$\sum_{n \leqslant x} d_k(n) = x \sum_{i=0}^{k-1} a_i^{(k)} (\log x)^{k-1-i} + O\left[x^{1-1/k}\right]. \qquad \ldots(4.1)$$

Now by Lemma 2.2

$$\sum_{n \leqslant x} d_k(n)\, \omega(n) = \sum_{i=1}^{k} (-1)^{i-1} \left[\begin{matrix} k \\ i \end{matrix}\right] \Big\{ \sum_{p^i n \leqslant x} d_k(n) \Big]$$

$$= \left[\begin{matrix} k \\ i \end{matrix}\right] \Sigma_1 + \sum_{i=2}^{k} (-1)^{i-1} \left[\begin{matrix} k \\ i \end{matrix}\right] \Sigma_i \qquad \ldots(4.2)$$

say.

For $i \geqslant 2$, we have by (4.1)

$$\sum\nolimits_i = \sum_{p \leqslant x^{1/i}} \sum_{n \leqslant x/p^i} d_k(n) = \sum_{p \leqslant x^{1/i}} \left\{ \frac{x}{p^i} \sum_{j=0}^{k-1} a_j^{(k)} \left[\log \frac{x}{p^i}\right]^{k-1-j} + O\left\{\left[\frac{x}{p^i}\right]^{1-1/k}\right]\right\}$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{p \leqslant x^{1/i}} \frac{1}{p^i} \left\{ \sum_{r=0}^{k-1} (-1)^r i^r \left[\begin{matrix} k-1-j \\ r \end{matrix}\right] (\log x)^{k-1-j-r} \right.$$

$$\left. \times (\log p)^r \right\} + O\left[x^{1-1/k} \sum_{p \leqslant x^{1/i}} p^{-i(1-1/k)}\right]$$

(*equation continued on p.* 1000)

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{r=0}^{k-1-j} (-1)^r i^r \begin{bmatrix} k-1-j \\ r \end{bmatrix} (\log x)^{k-1-j-r} \left\{ \sum_{p \leqslant x^{1/i}} \frac{\log p)^r}{p^i} \right\}$$

$$+ O\,[x^{1-1/k} \log\log x]$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{r=0}^{k-1-j} (-1)^r i^r \begin{bmatrix} k-1-j \\ r \end{bmatrix} (\log x)^{k-1-j-r} \left\{ \sum_{p} \frac{(\log p)^r}{p^i} \right.$$

$$\left. + O\left[\frac{(\log x)^r}{x^{1-1/i}} \right] \right\} + O\,[x^{1-1/k} \log\log x]$$

$$= x \sum_{j=0}^{k-1} D_j\, (\log x)^{k-1-j} + O\,(x^{1-1/k} \log\log x] \qquad \ldots(4.3)$$

for certain constants D_j, $0 \leqslant j \leqslant k-1$.

For the estimation of Σ_1 we use Dirichlet's hyperbola method.

$$\Sigma_1 + \sum_{pn \leqslant x} d_k(n) = \sum_{\substack{pn \leqslant x \\ n < \sqrt{x}}} d_k(n) + \sum_{\substack{pn \leqslant x \\ n \leqslant \sqrt{x}}} d_k(n) - \sum_{pn \leqslant x} d_k(n) - \sum_{p \leqslant \sqrt{x}, n \leqslant \sqrt{x}} d_k{}'n)$$

$$= \sum\nolimits_1^{(1)} + \sum\nolimits_1^{(2)} - \sum\nolimits_1^{(3)}, \qquad \ldots(4.3)$$

say.

Now it is known by elementary methods that for every $C > 0$

$$\sum_{n \leqslant x} \frac{1}{P} \log\log x + B + O\left[\frac{1}{(\log x)\, c} \right] \qquad \ldots(4.5)$$

where B is a constant. Hence

$$\sum\nolimits_1^{(1)} = \sum_{pn \leqslant x, p \leqslant \sqrt{x}} d_k(n) = \sum_{p \leqslant \sqrt{x}} \sum_{n \leqslant x/p} d_k(n)$$

$$= \sum_{p \leqslant \sqrt{x}} \left\{ \frac{x}{P} \sum_{j=0}^{k-1} a_j^{(k)} \left[\log \frac{x}{P} \right]^{k-1-j} + O\left[\left[\frac{x}{P} \right]^{1-1/k} \right] \right\}$$

$$= \sum_{j=0}^{k-1} a_j^{(k)} \sum_{p \leqslant n\sqrt{x}} \frac{x}{P} \sum_{r=0}^{k-1-j} (-1)^r \begin{bmatrix} k-1-j \\ r \end{bmatrix} (\log x)^{k-1-j-r} (\log P)^r$$

$$+ O\left[x^{1-1/2k} \right]$$

(*equation continued on p. 1001*)

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{j=0}^{k-1-j} (-1)^r \left[\begin{matrix} k-1-j \\ r \end{matrix} \right] (\log x)^{k-1-j-r} \sum_{p \leqslant \sqrt{x}} \frac{(\log P)^r}{p}$$

$$+ O\left[x^{1-1/2k} \right]$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} (\log x)^{k-1-j} \left\{ \log \log x - \log 2 + B + O\left(\frac{1}{(\log x)^c} \right) \right\}$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{r=1}^{k-1-j} (-1)^r \left[\begin{matrix} k-1-j \\ r \end{matrix} \right] (\log x)^{k-1-j-r} \left\{ \frac{(\log x)^r}{2^r . r} + E_r \right.$$

$$\left. + O\left[\frac{1}{(\log x)^c} \right] \right\} + O\left[x^{1-1/2k} \right] \qquad \ldots(4.6)$$

$$= x \log \log x \sum_{j=0}^{k-1} a_j^{(k)} (\log x)^{k-1-j} + x \sum_{j=0}^{k-1} b_j^{(k)} (\log x)^{k-1-j}$$

$$+ O\left[\frac{x}{(\log p^{c+1+k}} \right].$$

For the estimations of $\sum_1^{(2)}$ and $\sum_1^{(3)}$ we write $D_k(x) = \sum_{n \leqslant x} d_k(n)$ and $\Delta_k(x)$

$= D_k(x) - x \sum_{j=0}^{k-1} a_j^{(k)} (\log x)^{k-1-j}$. Then

$$\Sigma_1^{(3)} = \left[\sum_{p \leqslant \sqrt{x}} 1 \right] \left[\sum_{n \leqslant \sqrt{x}} d_k(n) \right] = \pi(\sqrt{x}) D_k(\sqrt{x})$$

$$= \left[\mathrm{Li}(\sqrt{x}) + O\left[\frac{\sqrt{x}}{(\log x)^c} \right] \right] D_k(\sqrt{x})$$

$$= \mathrm{Li}(\sqrt{x}) D_k(\sqrt{x}) + O\left[x/(\log x)^{c+1-k} \right] \qquad \ldots(4.7)$$

for any $c > 0$, by the prime number theorem. Here $\mathrm{Li}\; x = \int_2^x \frac{dt}{\log t}$. Also

$$\sum_1^{(2)} = \sum_{n \leqslant \sqrt{x}} d_k(n) \sum_{p \leqslant x/n} 1 = \sum_{n \leqslant \sqrt{x}} d_k(n) \left\{ \mathrm{Li}\left[\frac{x}{n} \right] + O\left[\frac{x/n}{(\log x/n)^c} \right] \right\}$$

$$= \sum_{n \leqslant \sqrt{x}} d_k(n)(n)\, \mathrm{Li}\left[\frac{x}{n} \right] + O\left[\frac{x}{(\log x/n^{c-k}} \right]. \qquad \ldots(4.8)$$

However, by partial summation

$$\sum_{n \leqslant \sqrt{x}} d_k(n)\, \mathrm{Li}\left[\frac{x}{n}\right] = Li(\sqrt{x})\, D_k(\sqrt{x}) + \int_1^{\sqrt{x}} \frac{D_k(t)}{\log(x/t)} \frac{Dx}{t^2}\, dt$$

$$= \mathrm{Li}(\sqrt{x})\, D_k(\sqrt{x}) + \frac{x}{\log x} \int_1^{\sqrt{x}} \frac{\Delta_k(t)}{t^2}\left[1 - \frac{\log t}{\log x}\right]^{-1} dt$$

$$+ x \int_1^{\sqrt{x}} \frac{1}{t \log x/t} \sum_{j=0}^{k-1} a_j^{(k)} (\log t)^{k-1-j}\, dt.$$

Now

$$\frac{x}{\log x} \int_1^{\sqrt{x}} \frac{\Delta_k(t)}{t^2}\left[1 - \frac{\log t}{\log x}\right]^{-1} dt = \frac{x}{\log x} \int_1^{\sqrt{x}} \frac{\Delta_k(t)}{t^2}\left\{1 + \frac{\log t}{\log x}\right.$$

$$\left. + \dots + \left[\frac{\log t}{\log x}\right]^{N-1} + O\left[\left[\frac{\log t}{\log x}\right]^N\right]\right\} dt$$

$$= \frac{x}{\log x} \sum_{j=0}^{N-1} \frac{1}{(\log x)^j} \int_1^{\infty} \frac{\Delta_k(t)(\log t)^j}{t^2}\, dt + O(x(\log x)^N).$$

Since $\Delta_k(x) = O[x^{1-1/k}]$ Further on writing u for x/t, we get

$$x \int_1^{\sqrt{x}} \frac{1}{t \log x/t} \sum_{j=0}^{k-1} a_j^{(k)} (\log t)^{k-1-j}\, dt = x \sum_{j=0}^{k-1} a_j^{(k)} \int_{\sqrt{x}}^{x} \frac{\{\log(x(u)\}^{k-1-j}}{x/u \log u}$$

$$\times \frac{x}{u^2}\, du$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \sum_{r=0}^{k-1-j} (-1)^r \left[\begin{matrix} k-1-j \\ l \end{matrix}\right] (\log x)^{k-1-j-r} \int_{\sqrt{x}}^{x} \frac{(\log u)^{r-1}}{u}\, du$$

$$= x \sum_{j=0}^{k-1} a_j^{(k)} \left\{(\log x)^{k-1-j} (\log 2) + \sum_{r=1}^{k-1-j} (-1)^r \left[\begin{matrix} k-1-j \\ r \end{matrix}\right]\right.$$

$$\left. \times (\log x)^{k-1-r} \frac{(1-2^{-r})(\log x)^r}{r}\right\} = x \sum_{j=0}^{k-1} F_j (\log x)^{k-1-j}$$

for certain constants F_j, $0 \leqslant j \leqslant k-1$. Thus with a large C, from (4.2) through (4.8) and the above, we obtain the theorem.

Remark : 4.1: The case $k = 2$ of Theorem 4.1 was earlier attempted by De Koninck and Mercier[6]. See also (De Koninck and Ivić[4], Chapter 9). However, there is a slip, as pointed out by Sitaramachandrarao[16]. Recently, Ivić[14], in addition to correcting this, gave an analytic proof of Theorem 4.1 and several other results. Also, Ivić raised the problem of proving Theorem 4.1 by an elementary method.

We state the following results that can be proved by the methods of this paper. The set of constants A, B, C, D and $E_0, E_1, \ldots, E_{N-1}$ may vary with the asymptotic formula. For each positive integer N, we have

$$\sum_{n\leqslant x} 2^{\omega}(n)\,\omega(n) = Ax(\log x)(\log\log x) + Bx\log x + Cx\log\log x$$

$$+ Dx + x\sum_{i=0}^{N-1}\frac{E_i}{(\log x)^i} + O\left[\frac{x}{(\log x)^N}\right]$$

$$\sum_{n\leqslant x}\mu(n)\,\omega(n) = x\sum_{i=2}^{N}\frac{E_{i-1}}{(\log x)^i} + O\left[\frac{x}{(\log x)^{N+1}}\right]$$

$$\sum_{n\leqslant x}\phi(n)\,\omega(n) = Ax^2\log\log x + Bx^2 + x^2\sum_{i=1}^{N-1}\frac{E_i}{(\log x)^i}$$

$$+ O\left[\frac{x^2}{(\log x)^N}\right]$$

and

$$\sum_{n\leqslant x}\sigma(n)\,\omega(n) = Ax^2\log\log x + Bx^2 + x^2\sum_{i=1}^{N-1}\frac{E_i}{(\log x)^i}$$

$$+ O\left[\frac{x^2}{(\log x)^N}\right].$$

Some of these also appear in Ivić[14]. Finally, we note that for estimating sums of the form $\sum_{n\leqslant x} f(n)\,g(n)$ where $f(n)$ is multiplicative and $g(n)$ is additive, the method of this paper can be used when $g(n)$ is an arbitrary additive function while it appears that the analytic method outlined in Ivić[14] can be used only when $g(n)$ is a prime-independent additive function.

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ON THE PROBLEM OF THREE GRAVITATING TRIAXIAL RIGID BODIES

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In this paper we consider the problem of three triaxial rigid bodies formulated in Delaunay and Andoyer variables. Taking certain constant values for the energy corresponding to the approximation of three point masses, we can obtain the effect of interaction of bodies due to the non-sphericity on the Delaunay and Andoyer variables and hence in these cases the general motion of the bodies in the space can be determined.

1. INTRODUCTION

When we are dealing with triaxial rigid bodies instead of mass points, the problem becomes even more complicated. Aksenov[1,2] obtained solutions to the planar double averaged elliptical restricted three body problem. More recently Sidlichovsky[3], considered the more general three body problem and obtained the solution of the equations of motion, assuming that the angular momentum of the close binary is much smaller than the angular momentum of the motion of the binary around a third body. Sidlichovsky[4], obtained the Hamiltonian of the problem of three rigid bodies in Delaunay and Andoyer variables averaging over the fast variables, applying various restrictive assumptions obtained different approximations of the problem. Duboshin[5] showsthat the problem of translatory-rotatory motion of three rigid bodies admits some particular solutions (Lagrangian and Eulerian solutions) when each body possesses axial symmetry and equatorial symmetry. Cid and Elipe[6] studied the plane motion of three rigid bodies besides the case of three axisymmetric ellipsoids. In the present work we consider the gravitating of three triaxial rigid bodies M_i ($i = 0, 1, 2$) such that the distance r of the bodies M_0 and M_1 is small compared to the distance r' of the body M_2 from the centre of inertia of the bodies M_0, M_1. Taking certain constant values for the energy corresponding to the approximation of three point masses, we can obtain the effect of interaction of bodies due to the non-sphericity on the Delaunay and Andoyer variables.

2. SYSTEM OF THREE RIGID BODIES

Let us describe the system of three triaxial rigid bodies M_0, M_1, M_2 with masses m_0, m_1, m_2 by Jacobi coordinates of their centres of inertia T_0, T_1, T_2. Let T be the center of inertia of the bodies M_0, M_1. Let x, y, z be the coordinates of T_1 in the

coordinate system S_0 with the origin in T_0 and axes parallel to the inertia system. Similarly, x', y', z' are the coordinates of T_2 in coordinate system S with the origin at T and the axis again parallel to the coordinate system. The rotational motion of each body is described by Andoyer variables[7] L_i, G_i H_i, l_i, g_i, h_i ($i = 0, 1, 2$).

Introducing

$$m = \frac{m_0\, m_1}{m_0 + m_1},\; m' = \frac{(m_0 + m_1)\, m_2}{m_0 + m_1 + m_2}$$

the kinetic energy T of the system is obtained as

$$T = \tfrac{1}{2}\, m\, (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \tfrac{1}{2}\, m'\, (\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2)$$

$$+ \sum_{k=0}^{2} \frac{1}{2} \left(\frac{\sin^2 l_k}{A^{(k)}} + \frac{\cos^2 l_k}{B^{(k)}} \right) (G_k^2 - L_k^2) + \frac{L_k^2}{2C^k} \qquad \ldots(1)$$

where $A^{(k)}$, $B^{(k)}$ and $C^{(k)}$ are the principal moments of intertia of the kth body. The potential energy consists of three terms

$$U = U_{01} + U_{02} + U_{03} \qquad \ldots(2)$$

where U_{ik} is the potential energy of the gravitational interaction of the ith and kth bodies. Only a part U' of (2) will be taken into account in the unperturbed Hamiltonian

$$U' = -f\, \frac{m_0\, m_1}{r} - f\, \frac{(m_0 + m_1)}{r'}\, m_2$$

where r is the distance $\overline{T_0\, T_1}$, r' is the distance $\overline{T\, T_2}$, and f is the gravitational constant. Introducing Delaunary variables $L, G, H, l, g, h, L', G', H'\, l', g', h'$ so that their relation with the Kepler elements a, a', e, e', I, I' are given in Sidlichovsky[4].

Let us assume that the mass m_2 is so distance from the orbiting system M_0, M_1 that its attraction may be approximation by that of a mass point. The Hamiltonian H^* of the problem after averaging over the fast variables l, l, l_0, l_1, g_0 and g_1 is given in Sidlichovsky[4]. The equation of motion can be written in the canonical form :

$$\frac{d}{dt}\,(l, g, h, l', g', h', l_i, g_i, h_i) = \frac{\partial H^*}{\partial\, (L, G, H, L', G', H'\, L_i, G_i, H_i)} \qquad \ldots(3)$$

$$\frac{d}{dt}\,(L, G, H, L', G'\, H', L_i, G_i, H_i) = -\, \frac{\partial H^*}{\partial\, (l, g, h, l', g', h', l_i, g_i, h_i)}$$

$$H^* = H_0^* + H_1^* \quad (i = 0, 1)$$

where H_0^*, H_1^* are unperturbed and perturbed Hamiltonian which defined in Sidlichovsky[4].

3. Approximation Integrals of the Equations of Motion

Now, to integrate the equations of motion we use the method of small parameter β which depend on the difference between the moments of inertia. In the zero approximation we take $\beta = 0$ ($A^{(i)} = B^{(i)} = C^{(i)}$) the Hamiltonian can be written in the form

$$R = -\frac{\mu^2 m^3}{2L^2} - \frac{\mu'^2 m'^3}{2L'^2} + R_{20}$$

where R_{20} (Sidlichovsky[4]), is the part corresponding to the approximation of three point masses. In the zero approximation the elements of motion depend on the energy constant R which are given in Sidlichovsky[3] we shall study three cases in the first approximation :

(a) $R = R_2 = \frac{1}{4}\,[6S + 5r - (135rS)^{1/2}],\ S \leqslant 3/5\,r$

$G^2 = (5/3\,rS)^{1/2} = \text{const.}$

(b) $R = R_3,\ S > 3/5\,r$

(c) $R = R_4 = \frac{1}{8}\,(3S - 5r)\ S < r$

$G^2 = S = \text{const.}$

where

$$S = (G' - C)^2,\ G' \geqslant G$$

$$r = L^2,\ H + H' = C = \text{const.}$$

In the first approximation ($\beta \neq 0$) the angles h, h', h_l and their conjugates are the only quantities which may be possibly affected by the non-sphericity of the bodies. The perturbation of these elements can be obtained from the following equations :

$$\partial h = \int_{t_0}^{t} \left(\frac{\partial H_1'^*}{\partial H}\right)_0 dt,\quad \delta h' = \int_{t_0}^{t} \left(\frac{\partial H_1^*}{\partial H'}\right)_0 dt$$

$$\delta h_l = \int_{t_0}^{t} \left(\frac{\partial H_1^*}{\partial H_l}\right) dt,\quad \delta H = -\int_{t_0}^{t} \left(\frac{\partial H_1^*}{\partial h}\right)_0 dt \qquad \ldots(4)$$

$$\partial H' = -\int_{t_0}^{t} \left(\frac{\partial H_1^*}{\partial b_l}\right)_0 dt,\quad \delta H_l = -\int_{t_0}^{t} \left(\frac{\partial H_1'^*}{\partial h_l}\right) dt$$

$$H_1'^{*} = H_1^{*} - R_{20}$$

where the subscript 0 refers to the values calculated in the zero approximation.

We shall obtain the perturbation (4) for the cases (*a*), (*b*), (*c*) as explicit functions of the time *t*. The expressions obtained are given by :

$$\delta H = -\sum_{i=0}^{1} \frac{3}{32} \frac{\mu^4 m^7}{m_i L_0^3 G_0^3} \left(\frac{2C^{(i)} - A^{(i)} - B^{(i)}}{\alpha}\right) \left(\frac{3L_{i0}^2}{G_{i0}^2} - 1\right)$$

$$\times \{\sin^2 I_0 \sin^2 I_i^0 \cos 2 [h_{i0} - \alpha (t - t_0)] - \sin 2 I_0 \sin 2I_{i0}$$

$$\cos [h_{i0} - \alpha (t - t_0)]\} \qquad \ldots(5)$$

$$\delta H' = -\frac{3}{32} \frac{\mu'^4 m'^7}{(m_0 + m_1) L_0'^3 G'^3} \sum_{i=0}^{1} \frac{(2C^{(i)} - A^{(i)} - B^{(i)})}{\alpha} \left(\frac{3L_{i0}^2}{G_{i0}^2} - 1\right)$$

$$\times \{\sin^2 I_0 \sin^2 I_0' \cos 2 [h_{i0} - \alpha (t - t_0)] + \sin 2I_0 \sin 2I_0'$$

$$\times \cos [h_{i0} - \alpha (t - t_0)]\} \qquad \ldots(6)$$

$$\delta H_i = \frac{3}{32} \left(\frac{2C^{(i)} - A^{(i)} - B^{(i)}}{\alpha}\right) \left(\frac{3L_{i0}^2}{G_{i0}^2} - 1\right) \left\{\frac{\mu'^4 m'^7 \sin 2I_0'}{(m_0 + m_1) L_0'^3 G_0'^3}\right.$$

$$+ \frac{\mu^4 m^7 \sin 2I_0}{m_i L_0^3 G_0^3} \sin 2I_i \cos (h_{i0} - \alpha (t - t_0))$$

$$\left. + \frac{\mu'^4 m'^7 \sin 2I_0'}{(m_0 + m_1) L_0'^3 G_0'} + \frac{\mu^4 m^7 \sin^2 I_0}{m_i L_0^3 G_0^3} \sin^2 I_{i0} \cos 2 (h_{i0} - \alpha (t - t_0))\right\}$$

$$\ldots(7)$$

and

$$\delta h = \sum_{i=0}^{1} \frac{3}{64} \frac{\mu^4 m^7}{m_i L_0^3 G_0^2 \sqrt{G^2 - H_0^2}} (2C^{(i)} - A^{(i)} - B^{(i)}) \left(\frac{3L_{i0}^2}{G_i^2} - 1\right)$$

$$\times \{[2\sin 2I_0 (3 \cos^2 I_{i0} - 1)] t + \frac{1}{\alpha} [\sin 2I_0 \sin^2 I_{i0}$$

$$\sin 2 (h_{i0} - \alpha (t - t_0)) - 2\cos 2I_0 \sin 2I_{i0} \sin (h_{i0} - \alpha (t - t_0))]\}$$

$$\delta h' = \frac{3}{64} \frac{\mu'^4 m'^7}{(m_0 + m_1) L_0'^3 G_0^3 \sqrt{G_0'^3 - H_0'^2}} \sum_{i=0}^{1} (2C^{(i)} - A^{(i)} - B^{(i)})$$

$$\times \left(\frac{3L_{i0}^2}{G_{i0}^2} - 1\right) \{[2\sin 2I_0' (3\cos^2 I_{i0} - 1)] t + \frac{1}{\alpha}$$

$$[\sin 2I_0' \sin^2 I_{i0} \sin 2 (h_{i0} - \alpha (t - t_0))$$

$$+ 2 \cos 2I_0' \sin 2I_{i0} \sin (h_{i0} - \alpha (t - t_0))]\}$$

$$\delta h_i = \frac{3}{64} \frac{(2C^{(i)} - A^{(i)} - B^{(i)})}{\sqrt{G_{i0}^2 - H_{i0}^2}} \left(\frac{3L_{i0}^2}{G_{i0}^2} - 1\right) \left\{2\sin 2I_{i0} \left[\frac{\mu^4 m^7 (3\cos^2 I_0 - I)}{m_i L_0^3 G_0^3}\right.\right.$$

$$\left. + \frac{\mu'^4 m'^7 (\cos^2 I' - 1)}{(m_0 + m_1) L_0'^3 G_0'^3}\right] t + \frac{1}{\alpha} [(\sin^2 I_0' + \sin^2 I_0) \sin 2I_{i0}$$

$$\times \sin 2 (h_{i0} - \alpha (t - t_0)) + 2 (\sin 2I_0 - \sin 2I_0) \cos 2I_i$$

$$\left.\times \sin (h_{i0} - \alpha (t + t))]\right\}$$

where α be determined from zero approximation, for the cases (*b*), (*c*) equal to

$$\frac{3}{16} \frac{f m_2 m a_0^2 (1 - e_0'^2)^{3/2}}{a_0'^3 \sqrt{G_0^2 - H_0'^2}} (\sin 2I_0 \cos 2I_0' + \sin 2I_0' \cos 2I_0)$$

and for the case (*a*) equal to

$$\frac{3}{16} \frac{f m_2 m a_0^2 (1 - e_0'^2)^{-3/2} (1 - 5 e_0'^2 \cos 2g_0)}{a_0'^3 \sqrt{G_0^2 - H_0'^2}}$$

$$(\sin 2I_0 \cos 2I_0' + \sin 2I_0' \cos 2I_0).$$

Thus, the perturbed inclinations of the orbital planes with respectt o the reference plane change according to (5), (6) and the perturbed orientation of the rotational angular momentum $\mathbf{G}_i$ ($i = 0, 1$) in space can be determined from (7). The results show that there is no effect from the perturbed Hamiltonian due to the non-sphericity of the

bodies on the eccentricities e, e' and the semimajor axes a, a' of orbits M_1, M_2. Also, the rotational angular momentum G_i and its orientation in the body M_i ($i = 0, 1$) remain unaltered.

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ON A PARTICULAR INITIAL VALUE PROBLEM, WITH AN APPLICATION IN RESERVIOR ANALYSIS

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In this paper, existence, uniqueness and continuous dependence of solutions to direct and inverse problems associated with a particular Cauchy problem for an ordinary differential equation are investigated. Such problems arise in the study of the water movement about a gas-storage reservoir of aquifer type when the corresponding influence functions degenerate.

1. INTRODUCTION

Consider a pair of real-valued finite functions $p(t)$ and $v(t)$ defined on the closed interval $I := [0, 1]$ such that the intermediate function

$$\Omega(t) := v(t)/p(t) \text{ in } I, \Omega_0 := v_0/p_0 \qquad \ldots(1)$$

satisfies the initial value problem

$$c \frac{d\,\Omega(t)}{dt} = p(t) - p_0 \text{ in } I, \Omega(0) = \Omega_0, c > 0. \qquad \ldots(2)$$

Let

$$p(0) = p_0, p(t) > 0 \text{ in } I \qquad \ldots(3)$$

and

$$v(0) = v_0, v(t) > 0 \text{ in } I \qquad \ldots(4)$$

where c, p_0 and v_0 are assumed to be fixed positive values throughout this paper. Moreover, denote by

$D_p := \{p \in C[0, 1] : p(t) \text{ fullfil } (3)\}$, $D_v := \{v \in C[0, 1] : v(t) \text{ fulfil } (4)\}$ and $D_\Omega := \{\Omega \in C^1[0, 1] : \Omega(0) = \Omega_0\}$ the subsets of admissible elements considered below. Often it will be useful to prescribe upper and lower bounds on v :

$$0 < v_{\min} \leqslant v(t) \leqslant v_{\max} < \infty \text{ in } I. \qquad \ldots(5)$$

Therefore, we additionally introduce the set

$\widetilde{D}_v := \{v \in D_v : v(t) \text{ fulfil (5)}\}$. Throughout this paper, $\|p\| := \max_{t \in I} |p(t)|$ and $\|P\| := \max_{\|v\|=1} \|Pv\|$ denote the norms of an element $p \in C[0, 1]$ and of a bounded linear operator $P : C[0, 1] \to C[0, 1]$, respectively.

If the water movement about gas-storage reservoirs of aquifer type is modelled by the influence function method[2], the time-varying dependence between reservoir pressure p and volume of gas v may be expressed by the Volterra-Stieltjes convolution integral equation

$$\int_0^t \frac{d\,\Omega(\tau)}{d\tau}\, d\,F(t-\tau) = p_0 - p(t) \text{ in } I. \qquad \ldots(6)$$

Here, the gas-occupied pore volume Ω satisfies eqn. (1). The continuous monotonically increasing influence function $F(t)$, with $F(0) = 0$ and $F(1) = c > 0$, possesses a continuous differentiable convex derivative $dF(t) \mid dt \leqslant 0$ whenever $t > 0$ and expresses the reservoir pressure response to a unit rate of water influx. An analysis of direct and inverse problems associated with eqns. (1) and (6) is given by the author[4] for the case that $dF(t)/dt \in C^1[0, 1]$. This paper intends to give an overview of the extremal situation characterized by degenerating the influence function into a Heaviside form

$$F(t) = \begin{cases} 0 & \text{if } t = 0 \\ c & \text{if } t > 0 \end{cases} \qquad \ldots(7)$$

This case is just described by eqns. (1) and (2).

2. Formulation of Problems

There are four tasks according to the initial value problem (1)-(2) which will be of our interest in the sequel :

(P1) — Find $p \in D_p$ when $c > 0$, $p_0 > 0$, $v_{\max} \geqslant v_{\min} > 0$ and $v \in \widetilde{D}_v$ are given !

(P2) — Find $v \in D_v$ when $c > 0$, $v_0 > 0$ and $p \in D_p$ are given !

(P3) — Find $c > 0$ when $v_0 > 0$, $v_1 := v(1)$ and $p \in D_p$ are given !

(P4) — Find $v_0 > 0$ when $c > 0$, $p_0 > 0$, $p(t) > 0$ for a certain sub-interval $t \in [t_1, t_2]$ and $w(t) := v(t) - v_0$ in $t \in [t_1, t_2]$, $0 \leqslant t_1 < t_2 \leqslant 1$ are given !

The prediction problem (P1) is of direct nature, whereas the control problem (P2) and the identification problems (P3) and (P4) belong to the class of inverse problems[3]. Indeed, deviations of $p(t)$ from the initial value p_0 are caused by deviations of values $v(\tau)$, $0 < \tau < t$ from the initial value v_0.

We are going to study the existence, uniqueness and stability of solutions to the problems introduced above, where a particular a priori bound principle expressed by Lemma 2.1 is essentially helpful.

Lemma 2.1—Let for given $v_{max} \geqslant v_{min} > 0$, $p \in D_p$ and $v \in \widetilde{D}_v$ satisfy the eqns. (1) and (2). Then we have

$$0 < \Omega_{min} := \frac{v_{min}}{v_0} \Omega_0 \leqslant \Omega(t) \leqslant \Omega_{max} := \frac{v_{max}}{v_0} \Omega_0 \text{ in } I \quad \ldots (8)$$

and

$$0 < p_{min} := \frac{v_{min}}{v_{max}} p_0 \leqslant p(t) \leqslant p_{max} := \frac{v_{max}}{v_{min}} p_0 \text{ in } I. \quad \ldots (9)$$

PROOF : If $p \in D_p$ and $v \in \widetilde{D}_v$ satisfy eqn. (2), then it follows $\Omega \in D_\Omega$. Now let the function $\Omega(t)$ attain its absolute maximum over I at the point $\hat{t} > 0$. This implies $d\,\Omega(\hat{t})/dt > 0$ and thus $p(\hat{t}) \geqslant p_0$. Consequently, $\Omega(t) \leqslant \Omega(\hat{t}) = v(\hat{t})/p(\hat{t}) \leqslant v_{max}/p_0 = (v_{max}/v_0)\,\Omega_0 =: \Omega_{max}$ in I and $p(t) \geqslant v(t)/\Omega(t) \geqslant v_{min}/\Omega_{max} = (v_{min}/v_{max})\,p_0$. If otherwise $\Omega(t) \leqslant \Omega_0$ in I, then $\Omega(t) \leqslant \Omega_{max}$ holds evidently. In an analogous manner one obtains the remaining two bounds of formulae (8) and (9) from $d\,\Omega(\hat{\hat{t}})/dt \leqslant 0$ when $\hat{\hat{t}} > 0$ denotes the minimum point of $\Omega(t)$ over I. This proves the Lemma.

Note that the bounds Ω_{min}, Ω_{max}, p_{min} and p_{max} are independent of the constant $c > 0$ in eqn. (2). Thus, the subsequent considerations can essentially be confined to the subset $\widetilde{D}_p := \{p \in D_p : p(t) \text{ fulfil } (9)\}$ of D_p. Let us complete this paragraph with the remark that due to Lemma 2.1 a steady state of the form $v(t) \equiv v_0$ in I implies a steady state $p(t) \equiv p_0$ in I. The converse is immediately obtained from eqn. (2).

3. On The Well-Posedness of Problem (P1)

This study proves the problem (P1) to be uniquely solvable for all prescribed data and presents a stability theorem regarding perturbations of $v \in \widetilde{D}_v$.

Theorem 3.1—For any given $c > 0$, $p_0 > 0$, $v_{max} \geqslant v_{min} > 0$ and $v \in \widetilde{D}_v$, the problem P(1) is uniquely solvable in $p \in D_p$. Moreover, this solution belongs to the subset $\widetilde{D}_p$.

PROOF : We consider the initial value problem

$$\frac{d\,\Omega(t)}{dt} = \frac{1}{c}\left(\frac{v(t)}{\Omega(t)} - p_0\right), \quad -1 \leqslant t \leqslant 1,\ \Omega(0) = \Omega_0 \quad \ldots (10)$$

where the function v is extended to negative arguments $t < 0$ by $v(t) \equiv v_0$. Provided the problem (10) is uniquely solvable in $\Omega \in C^1[0, t_0]$, $t_0 > 0$, then we have because

of Lemma 2.1 $\Omega_{\min} \leqslant \Omega(t_0)$. Now set $\alpha := 1 - t_0$, $\beta := \Omega_{\min}/2$ and $\gamma := \frac{1}{c}\left(\frac{2}{\Omega_{\min}} v_{\max} - + p_0\right)$. Obviously,

$$\left|\frac{1}{c}\left(\frac{v(t)}{\Omega(t)} - p_0\right)\right| \leqslant \gamma \text{ and } \left|\frac{1}{c}\left(\frac{v(t)}{\Omega_2} - p_0\right) - \frac{1}{c}\left(\frac{v(t)}{\Omega_2} - p_0\right)\right|$$

$$\leqslant \frac{4\, v_{\max}}{c\, \Omega^2_{\min}} - \|\Omega_1 - \Omega_2\|$$

whenever $|t - t_0| \leqslant \alpha$, $|\Omega_1 - \Omega(t_0)| \leqslant \beta$, $|\Omega_2 - \Omega(t_0)| \leqslant \beta$ and $|\Omega(t) - \Omega(t_0)| \leqslant \beta$. By employing the Picard-Lindelöf theorem[5] we thus obtain a uniquely determined continuously differentiable solution $\Omega(t)$ in $|t - t_0| \leqslant \min(\alpha, \beta/\gamma)$. Since the quotient β/γ is independent of t_0, there is exactly one solution $\Omega \in D_\Omega$ satisfying the formulae (8) and (10). This implies the existence of a uniquely determined positive solution $p(t) = v(t)/\Omega(t)$ in I. As a consequence of Lemma 2.1 this solution belongs to $\widetilde{D}_p$. This completes the proof.

Because of Theorem 3.1 there exists a uniquely determined operator $P : \widetilde{D}_v \subset C[0, 1] \to \widetilde{D}_p \subset C[0, 1]$ that transforms $v \in \widetilde{D}_v$ into the associated solution $p := Pv \in \widetilde{D}_p$ of problem (P1).

Theorem 3.2—Let $p^{(1)} := Pv^{(1)}$ and $p^{(2)} := Pv^{(2)}$ denote the solutions of problem (P1) according to $v^{(1)} \in \widetilde{D}_v$ and $v^{(2)} \in \widetilde{D}_v$, respectively. Then we obtain a Lipschitz condition of the form

$$\|p^{(1)} - p^{(2)}\| \leqslant \frac{p^2_{\max}}{p_{\min}\, v_{\min}} \left(1 + \frac{p^2_{\max}}{v_{\min}}\right) \|v^{(1)} - v^{(2)}\|. \quad \ldots(11)$$

Proof : We introduce the operator $H : D_p \times D_v \subset (C[0, 1])^2 \to C[0, 1]$ by the formula $(H(p, v))(t) := \frac{v_0}{p_0} - \frac{v(t)}{p(t)} \int\limits_0^t (p(\tau) - p_0)\, d\tau$ in I. There exist partial Fréchet derivatives

$$(\partial_p H(\hat{p}, \hat{v})\, p)(t) = \frac{\hat{v}(t)}{\hat{p}^2(t)}\, p(t) + \int\limits_0^t p(\tau)\, d\tau$$

and

$$(\partial_v H(\hat{p}, \hat{v})\, v)(t) = -\frac{v(t)}{\hat{p}(t)}$$

at any point $(\hat{p}, \hat{v}) \in D_p \times D_v$ which are continuous in a neighbourhood of this point. It gets evident that the linear Volterra integral operator of the second kind

$\partial_p H(\hat{p}, \hat{v})$ has a uniformly bounded inverse satisfying the inequality

$$\|(\partial_p H(\hat{p}, \hat{v}))^{-1}\| \leqslant \frac{p^2_{\max}}{v_{\min}} \left(1 + \frac{p^2_{\max}}{v_{\min}}\right) \qquad \ldots(12)$$

whenever $\hat{p} \in \widetilde{D}_p$ and $\hat{v} \in \widetilde{D}_v$. Namely, $\dfrac{\hat{v}(t)}{\hat{p}^2(t)}\, p(t) + \displaystyle\int_0^t p(\tau)\, d\tau = q(t)$ in I is equivalent to

$$\frac{d\varphi(t)}{dt} + \frac{da(t)}{dt}\, \varphi(t) = \rho(t) \text{ in } I \qquad \ldots(13)$$

where

$$\varphi(t) := \int_0^t p(\tau)\, d\tau, \; \frac{da(t)}{dt} := \frac{\hat{p}^2(t)}{\hat{v}(t)}, \; a(0) := 0$$

and

$$\rho(t) := \frac{\hat{p}^2(t)}{\hat{v}(t)}\, q(t).$$

We can write the solution of the initial value problem (13) by the explicit formula[1]

$$\varphi(t) = \exp(-a(t)) \times \int_0^t \exp(a(\tau))\, \rho(\tau)\, d\tau.$$

This implies

$$p(t) = -\exp(-a(t))\, \frac{\hat{p}^2(t)}{\hat{v}(t)} \times \int_0^t \exp(a(\tau))\, \frac{\hat{p}^2(\tau)}{\hat{v}(\tau)}\, q(\tau)\, d\tau + \frac{\hat{p}^2(t)}{\hat{v}(t)} q(t)$$

and therefore the inequality (12). Moreover,

$\|\partial_v H(\hat{p}, \hat{v})\| \leqslant p_{\min}^{-1}$ if $\hat{p} \in \widetilde{D}_p$ and $\hat{v} \in \widetilde{D}_v$. Hence, as a consequence of the implicit function theorem[5] we obtain for the Fréchet derivative P' of P at the point $\hat{v} = \vartheta v^{(1)} + (1 - \vartheta)\, v^{(2)} \in \widetilde{D}_v$, $0 \leqslant \vartheta \leqslant 1$,

$$\|P'(\hat{v})\| \leqslant \|(\partial_p H(P\hat{v}, \hat{v}))^{-1}\|\, \|\partial_v H(P\check{v}, \hat{v})\| \leqslant \frac{p^2_{\max}}{p_{\min}\, v_{\min}} \left(1 + \frac{p^2_{\max}}{v_{\min}}\right).$$

Then the Lipschitz condition (11) results from the well-known remainder bound structure of Taylor series[5]. This proves the theorem.

4. Solution of Problems (P2) and (P3)

By integrating eqn. (2) one obtains

$$c\,(\Omega\,(t) - \Omega_0) = \int_0^t p\,(\tau)\,d\tau - p_0\,t \text{ in } I. \qquad \ldots(14)$$

From this formula there are derived explicit representations of the solution to problem (P2),

$$v\,(t) = p\,(t)\,\left\{\left(\Omega_0 - \frac{p_0}{c}\,t\right) + \frac{1}{c}\int_0^t p\,(\tau)\,d\tau\right\} \text{ in } I \qquad \ldots(15)$$

and provided $\Omega\,(1) \neq \Omega_0$ of the solution to problem (P3),

$$c = \frac{\int_0^1 p\,(t)\,dt - p_0}{\Omega\,(1) - \Omega_0}\,. \qquad \ldots(16)$$

Hence, problem (P2) is uniquely solvable in $v \in D_v$ if and only if the prescribed function $p \in D_p$ satisfies the inequality

$$\int_0^t p\,(\tau)\,d\tau > p_0\,t - \Omega_0\,c \text{ in } I. \qquad \ldots(17)$$

If otherwise (17) gets injured for some value $t \in I$, then (P2) possesses no solution. Evidently, the prescribed values $p\,(t) > 0$ in the subinterval $0 \leqslant t < \Omega_0\,c/p_0$ can never violate the inequality (17) since $p_0\,t - \Omega_0\,c$ is negative in this case. That means, arbitrary positive continuous functions $p\,(t)$ in $0 \leqslant t \leqslant t_0$, may be generated by appropriate positive functions $v\,(t)$ in $0 \leqslant t \leqslant t_0$ attaining the initial value $v\,(0) = v_0$ whenever $t > 0$ is sufficiently small.

Theorem 4.1—Let $c > 0$, $v_0 > 0$ and $p \in D_p$ be given so that the inequality (17) is satisfied. Then, problem (P2) is uniquely solvable in $v \in D_v$ and we have a local Lipschitz condition

$$\|v^{(1)} - v^{(2)}\| \leqslant \left(\max_{t\in I}\;\left|\,\Omega_0 - \frac{p_0}{c}\,t\,\right| + \frac{\|p^{(1)}\| + \|p^{(2)}\|}{c}\right) \cdot \|p^{(1)} - p^{(2)} \qquad \ldots(18)$$

where, for fixed positive values c, p_0 and v_0 $v^{(1)} \in D_v$ and $v^{(2)} \in D_v$ are solutions to problem (P2) according to prescribed data $p^{(1)} \in D_p$ and $p^2 \in D_p$, respectively.

Proof: The Lipschitz condition (18) follows from eqn. (15) using the triangle inequality. For all $t \in I$, we have

$$|\,v^{(1)}\,(t) - v^{(2)}\,(t)\,| \leqslant |\;\left((\Omega_0 - \frac{p_0}{c}\,t\right)(p^{(1)}\,(t)\,p^{(2)}\,(t))$$

(*equation continued on p.* 1017)

$$+ \quad \frac{1}{c} \left\{ p^{(1)}(t) \int_0^t p^{(1)}(\tau)\, d\tau + p^{(2)}(t) \int_0^t p^{(2)}(\tau)\, d\tau \right.$$

$$\leqslant | \Omega_0 - \frac{p_0}{c} t \| p^{(1)}(t) - p^{(2)}(t) | + \frac{1}{c} \left\{ p^{(1)}(t) \int_0^t \right.$$

$$+ \ | p^{(1)}(\tau) - p^{(2)}(\tau) |\, d\tau + | p^{(1)}(t) - p^{(2)} t |$$

$$\int_0^t p^{(2)}(\tau)\, d\tau \leqslant \left\{ | \Omega_0 - \frac{p_0}{c} t | + \frac{p^{(1)}(t) + p^{(2)}(t)}{c} \right.$$

$$\times \ | p^{(1)}(t) - p^{(2)}(t) | \, .$$

This implies the formula (19) and completes the proof.

Note that the inverse problem

(P2*) — Find $v \in D_v$ when $c > 0$ and $p \in D_p$ are given ! is not uniquely solvable since v_0 cannot be determined by the given data. Indeed, if formula (15) yields positive values $v(t)$ in I for a certain value Ω_J, then $v(t)$ also remain positive if Ω_0 attains larger values of arbitrary magnitude.

No we focus our attention on problem (P3). Apart from the steady state $v(t) \equiv v_0$, $p(t) \equiv p_0$ in I the requirement $\Omega(1) \neq \Omega_0$ is natural. Then the prescribed data determine a unique solution $c > 0$ whenever they are compatible with eqns. (1) and (2).

Theorem 4.2—Let, for fixed $v_0 > 0$ and $p_0 > 0$, $c^{(1)} > 0$ and $c^{(2)} > 0$ denote the solutions of problem P(3) according to prescribed data $p^{(1)} \in D_p$, $v^{(1)}(1) > 0$ and $p^{(2)} \in D_p$, $v^{(2)}(1) > 0$, respectively. Moreover, let

$$\Delta^{(1)} : = v^{(1)}(1)/p^{(1)}(1) - \Omega_0 \neq 0$$

and

$$\Delta^{(2)} : = v^{(2)}(1)/p^{(2)}(1) - \Omega_0 \neq 0.$$

Then we have

$$| c^{(1)} - c^{(2)} | \leqslant \frac{\left| \dfrac{v^{(1)}(1)}{p^{(1)}(1)} - \dfrac{v^{(2)}(1)}{p^{(2)}(1)} \right| \left| \displaystyle\int_0^1 p^{(1)}(\tau)\, d\tau - p_0 \right|}{| \Delta^{(1)} | \ | \Delta^{(2)} |} + \frac{\| p^{(1)} - p^{(2)} \|}{| \Delta^{(2)} |} \quad \ldots (19)$$

The triangle inequality is employed to establish the result of Theorem 4.2. Note that the estimation (19) reflects the various influence factors with respect to perturba-

tions in the data. For a stable recognition of the parameter c, first of all, the data must be chosen so that $|\Delta^{(1)}|$ and $|\Delta^{(2)}|$ are large enough. Evidently, $\Delta := \Omega(1) - \Omega_0 = 0$ is equivalent to $\int_0^1 p(t)\,dt = p_0$, i.e., p_0 is the mean value of $p(t)$ in I. If however $v(t) \geqslant v_0$ and $v(t) \not\equiv v_0$ in $[0, \epsilon]$ for all $\epsilon > 0$, then $\Omega(t) > \Omega_0$ for all $t > 0$. In order to avoid small values of Δ, it suffices to force a sufficiently rapid growth of $v(t)$ over the interval I.

5. Some Remarks on Problem (P4)

A nontrivial task of practical significance is the stable recovering of the initial value v_0 when noisy data of $p(t)$ and only increment values $w(t) = v(t) - v_0$ are available. We make some remarks on a special version of this class of problems expressed by (P4).

Let

$$\Omega^{(1)}(t) := \frac{v_0}{p_0} + \frac{1}{c}\int_0^t (p^{(1)}(\tau) - p_0)\,d\tau, \quad p^{(1)}(t) := \frac{v(t)}{\Omega^{(1)}(t)}$$

and

$$\Omega^{(2)}(t) := \frac{v_0 + \kappa}{p_0} + \frac{1}{c}\int_0^1 (p^{(2)}(\tau) - p_0)\,d\tau, \quad p^{(2)}(t) := \frac{v(t) + \kappa}{\Omega^{(2)}(t)}$$

hold for all $t \in I$ and a certain value $\kappa > 0$. Then we have

$$\frac{v(t_2) + \kappa}{p^{(2)}(t_2)} - \frac{v(t_2)}{p^{(1)}(t_2)} - \left(\frac{v(t_1) + \kappa}{p^{(2)}(t)} - \frac{v(t)}{p^{(1)}(t_1)}\right)$$

$$= \frac{1}{c}\int_{t_1}^{t_2} (p^{(2)}(t) - p^{(1)}(t))\,dt \quad \text{if } 0 \leqslant t_1 \leqslant t_2 \leqslant 1.$$

Hence,

$$\kappa\left(\frac{1}{p^{(2)}(t_2)} - \frac{1}{p^{(2)}(t_1)}\right) = \frac{1}{c}\int_{t_1}^{t_2} (p^{(2)}(t) - p^{(1)}(t))\,dt$$
$$+ (v_0 + w(t))\left(\frac{1}{p^{(2)}(t_1)} - \frac{1}{p^{(1)}(t_1)}\right)$$
$$+ (v_0 + w(t_2))\left(\frac{1}{p^{(1)}(t_2)} - \frac{1}{p^{(2)}(t_2)}\right),$$
$$0 \leqslant t_1 < t_2 \leqslant 1. \qquad \ldots(20)$$

If moreover $w(t) = 0$ in $t \in [t_1, t_2]$ and $p^{(2)}(t_1) \neq p^{(2)}(t_2)$, then we obtain a relative error estimation

$$\frac{\kappa}{v_0} \leqslant \frac{\left\{\frac{t_2 - t_1}{c} + \frac{2}{p_0^2}\left(1 + \frac{\|w\|}{v_{0*}}\right)\right\} \|p^{(2)} - p^{(1)}\|_*}{\left|\frac{1}{p^{(2)}(t_2)} - \frac{1}{p^{(2)}(t_1)}\right|},$$

$$\|p\|_* := \max |p(t)| \; t \in [t_1, t_2] \qquad \ldots(21)$$

Theorem 5.1—Let exist a solution $v_0 > 0$ to problem (P4). Then this solution is unique whenever

$$\frac{w(t) - w(t_1)}{t - t_1} \not\equiv \frac{p(t_1)(p(t_1) - p_0)}{c} \text{ in } t \in [t_1, t_2]. \qquad \ldots(22)$$

Proof : Provided that $p(t_1) \neq p(t_2)$ holds, the uniqueness of the solution to problem (P4) immediately follows from formula (20) by setting $p(t) = p^{(1)}(t) = p^{(2)}(t)$ in $t \in [t_1, t_2]$. Evidently, we obtain $\kappa = 0$ and therefore $v_0 > 0$ as a uiquely determinded value from the prescribed data. On the other hand, we can argue in the same manner whenever there is at least one value t satisfying $t_1 < t \leqslant (t_2)$ and $p\, t_1 \neq p(t)$. However, this requirement is only injured if $p(t_1) \equiv p(t)$ in $t \in [t_1, t_2]$. This identity implies

$$\Omega(t) = \Omega(t_1) + \frac{(t - t_1)}{c} - (p(t_1) - p_0)$$

i. e.,

$$v(t) = v(t_1) + \frac{p(t_1)(p(t_1) - p_0)(t - t_1)}{c}$$

and

$$\frac{w(t) \quad w(t_1)}{t - t_1} \equiv \frac{p(t)\, p(t_1) - p_0)}{c} \text{ in } t \in [t_1, t_2].$$

This provides the requirement (22) and completes the proof.

Finally, note that formula (20) also yields the continuous depenednce of solutions to problem (P4) upon the prescribed data whenever these data are compatible with eqns. (1) - (2) and satisfy the assumptions of Theorem 5.1.

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P-WAVE SCATTERING AT A COASTAL REGION IN A SHALLOW OCEAN

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The present problem can be idealised to study diffraction of compressional waves at a coastal region in a shallow ocean. The method of solution is the Wiener-Hopf technique. An exact solution is obtained in terms of Fourier integrals. The evaluation of the integrals along an appropriate contour in a complex plane results in reflected Rayleigh and scattered waves. Near the scatterer, the scattered field has the behaviour of a cylindrical wave and at distant points, it is a plane wave. The scattered field excites the particle motion at right angles to the free surface of the ocean and the horizontal component of the displacement is absent in the surface. Numerical computations for the scattered field in terms of the distance from the scatterer and for the Rayleigh waves versus depth below the free surface reveal a sharp fall for the former and a slow decay in case of the latter.

1. Introduction

The problem of diffraction of elastic waves by a rigid plane barrier in a liquid medium has been investigated by various authors. The effect of a vertical plane barrier, fixed in an infinitely deep sea, on normally incident surface waves was first considered by Ursell[1] dealing with a two-dimensional case. Faulkner[2] has extended Ursell's study to a three-dimensional case to consider the effect of a fixed vertical plane barrier of finite depth on obliquely incident surface waves. He has used the Wiener-Hopf technique[3] and has obtained approximate expressions for the transmission and reflection co-efficients. The problem of diffraction of compressional waves which strike obliquely at a vertical barrier of finite depth fixed in a liquid halfspace or in a liquid layer over a solid half-space has been discussed by Deshwal[4-6]. Mann and Deshwal[7] have studied the problem of Rayleigh wave scattering at the edge of a finite plane barrier in the surface of a shallow ocean over a solid halfspace using the technique of Wiener and Hopf. The scattered waves have the behaviour of reflected plane waves near the scatterer and of cylindrical waves at distant points.

The present paper deals with the problem of diffraction of compressional waves at a coastal region in a shallow ocean. At a coastal region, there is shallow ocean over the solid mantle of the earth on one hand and there are hard rocks on the other hand. The shallow ocean is assumed to be a liquid layer and the hard rock to be a rigid plane boundary. The region $x \leqslant 0, z \geqslant 0$ is a rigid medium in which there is

no movement of waves. In the region $x \leqslant 0$, $z \geqslant 0$, there is a shallow ocean in the form of a liquid layer of depth h occupying the region $x \geqslant 0$, $z \leqslant h$ lying over a solid half space $x \geqslant 0$, $z \geqslant h$.

2. Statement of the Problem

The two media lie on the side of positive z-axis (pointing vertically downward). The liquid layer and the solid halfspace occupy the regions $x \geqslant 0$, $0 \leqslant z \leqslant h$ and $x \geqslant 0$, $z \geqslant h$ respectively as in the Fig. 1. A two-dimensional compressional wave is

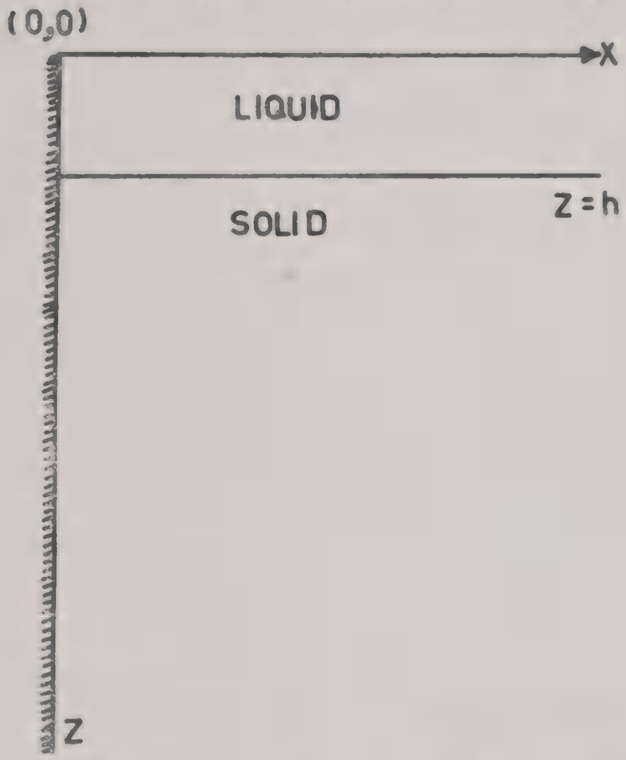

FIG. 1 Oceanic layer over solid mantle of the earth.

incident at the rigid plane boundary and the corner scatters the waves. The wave equations in the liquid and the solid are

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0, k = k_1 + ik_2 \qquad ...(1)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} + k'^2 \phi_1 = 0, k' = k_1' + ik_2' \qquad ...(2)$$

and a similar equation in ψ_1. k and k' are complex with their imaginary parts being small and positive. ϕ, ϕ_1 and ψ_1 are the displacement potentials in the liquid and the solid media respectively.

The incident potential satisfying the stress-free conditions is

$$\phi_i = D \sin (kz \sin \theta) \exp (- ikx \cos \theta) \qquad ...(3)$$

such that the total potential in the liquid layer is

$$\phi_t = \phi_l + \phi\,(x, z). \qquad \ldots(4)$$

The displacement components (u_1, w_1) in the solid medium in terms of potentials are given by

$$u_1 = \frac{\partial \phi_1}{\partial x} - \frac{\partial \psi_1}{\partial z}, \quad w_1 = \frac{\partial \phi_1}{\partial z} + \frac{\partial \psi_1}{\partial x}. \qquad \ldots(5)$$

Since there is no transverse wave in the liquid, the ψ-potential is absent and the displacement components (u, w) in the liquid layer are in terms of the derivatives of ϕ only.

3. Boundary Conditions

The conditions on the boundaries are :

(i) $\phi_t\,(x, z) = 0$, on $z = 0$, for all x $\qquad \ldots(6)$

(ii) $\dfrac{\partial \phi_t}{\partial x} = 0$, on $x = 0$, for $0 \leqslant z \leqslant h$ $\qquad \ldots(7)$

(iii) $\dfrac{\partial \phi_1}{\partial x} - \dfrac{\partial \psi_1}{\partial z} = 0$, on $x = 0$, for $z \geqslant h$ $\qquad \ldots(8)$

(iv) $\dfrac{\partial \phi_1}{\partial z} + \dfrac{\partial \psi_1}{\partial x} = 0$, on $x = 0$, for $z \geqslant h$ $\qquad \ldots(9)$

(v) $w = w_1$, $p_{zz} = (p_{zz})_1$, $(p_{zx})_1 = 0$, on $z = h$ $\qquad \ldots(10)$

for $x \geqslant 0$, where the subscript 1 denotes the entities belonging to the solid and p_{zz}, p_{zx} are the normal and shear stresses.

We define the Fourier transform

$$\bar{\phi}_+(p, z) = \int_0^\infty \phi\,(x, z)\, e^{ipx}\, dx, \; p = \alpha + i\beta \qquad \ldots(11)$$

and assume that, for given z,

$$|\,\phi\,| \sim D_1 \exp\,(-k_2\,|\,x\,|\,) \text{ as } |\,x\,| \to \infty \qquad \ldots(12)$$

where D_1 is a constant, then

$$|\,\bar{\phi}_+(p, z)\,| \leqslant \int_0^\infty |\,\phi\,(x, z)\,|\, e^{-\beta x}\, dx. \qquad \ldots(1\ \ )$$

On account of the assumption, $\bar{\phi}_+(p, z)$ is bounded at infinity only when $\beta > -k_2$. Therefore $\bar{\phi}_+(p, z)$ is regular in the region $\beta > -k_2$ of the complex p-plane. $\bar{\phi}_{1+}(p, z)$ and $\psi_{1+}(p, z)$ obey the same behaviour.

4. Solution of the Problem

Fourier transformation of (1) leads to

$$\frac{d^2}{dz^2}\,\bar{\phi}_+(p, z) - Y^2\,\bar{\phi}_+(p, z) = \left(\frac{\partial\phi}{\partial x}\right)_0 - ip\,(\phi)_0 \qquad \text{...(14)}$$

where

$$Y = \pm\,(p^2 - k^2)^{1/2}. \qquad \text{...(15)}$$

The sign before the radical is such that Re $(Y) \geqslant 0$ for all p. The subscript 0 denotes the value at $x = 0$. Let us change p to $-\,p$ in (14) and add the new equation to it to find

$$\frac{d^2}{dz^2}\,[\bar{\phi}_+(p, z) + \bar{\phi}_+(-\,p, z)] - Y^2\,[\bar{\phi}_+(p, z) + \bar{\phi}_+(-\,p, z)]$$

$$= D\,(2ik\cos\theta)\sin(kz\sin\theta). \qquad \text{...(16)}$$

The right hand side is the value of $(\partial\phi/\partial x)_0$ from (7) and the complete solution of this differential equation is

$$\bar{\phi}_+(p, z) + \bar{\phi}_+(-\,p, z) = c_1(p)\,e^{-Yz} + c_2(p)\,e^{Yz}$$

$$-\,\frac{(2ik\,D\cos\theta)\sin(kz\sin\theta)}{p^2 - k^2\cos^2\theta}. \qquad \text{.. (17)}$$

Let us utilize the condition (6) to find $c_2(p)$ in terms of $c_1(p)$. The condition results in

$$\bar{\phi}_+(p, 0) + \bar{\phi}_+(-\,p, 0) = 0. \qquad \text{...(18)}$$

Using (18) in (17) to have

$$\bar{\phi}_+(p, z) + \bar{\phi}_+(-\,p, z) = 2c_1(p)\sin hYz - \frac{(2ik\cos\theta)\,D\sin(kz\sin\theta)}{(p^2 - k^2\cos^2\theta)}. \qquad \text{...(19)}$$

Putting $z = h$ in (19) and in its derivative with respect to z and eliminating $c_1(p)$ between the resulting equations, we find

$$\bar{\phi}_+(p) + \bar{\phi}_+(-\,p) = \frac{1}{Y}\tanh Yh\;[\bar{\phi}'_+(p) + \bar{\phi}'_+(-p)]$$

$$-\,\frac{(2ik\cos\theta)\,D\sin(kh\sin\theta)}{(p^2 - k^2\cos^2\theta)}$$

$$+\,\frac{(2ik^2\sin\theta\cos\theta)\,D\cos(kh\sin\theta)}{(p^2 - k^2\cos^2\theta)}\cdot\frac{\tanh Yh}{Y}. \qquad \text{...(20)}$$

We have used the notations $\bar{\phi}_+(p)$, $\bar{\phi}'_+(p)$ for $\bar{\phi}_+(p, h)$ and $\bar{\phi}'_+(p, h)$. Similar notations will be used for the potentials in the solid medium.

Let us take the Fourier transform of the conditions (10) to get

$$\bar{\phi}'_+(p) + \frac{(ik \sin \theta) D \cos (kh \sin \theta)}{(p - k \cos \theta)} = \bar{\phi}'_{1+}(p) - ip \bar{\psi}_{1+}(p),$$

$$p \neq k \cos \theta \qquad \ldots(21)$$

$$-\lambda k^2 \left[\bar{\phi}_+(p) + \frac{iD \sin (kh \sin \theta)}{(p - k \cos \theta)}\right] = -\lambda_1 k'^2 \bar{\phi}_{1+}(p)$$

$$+ 2 \mu_1 (p^2 - k'^2) \bar{\phi}_{1+}(p) - 2 \mu_1 ip \bar{\psi}'_{1+}(p) \qquad \ldots(22)$$

and

$$-2ip \bar{\phi}'_{1+}(p) + (k''^2 - 2p^2) \bar{\psi}_{1+}(p) = 0, \text{ on } z = h, x \geqslant 0. \qquad \ldots(23)$$

λ_1, μ_1 are Lame's constants of the solid and λ is that of the liquid, k'' being the wave number for transverse waves in the solid. Taking the Fourier transformation of (2) and using the boundary conditions (8) and (9), we find

$$\frac{d^2}{dz^2} [\bar{\phi}_{1+}(p, z) - \bar{\phi}_{1+}(-p, z)] - Y_1^2 [\bar{\phi}_{1+}(p, z) - \bar{\phi}_{1+}(-p, z)] = 0,$$

$$Y_1 = \pm (p^2 - k'^2)^{1/2} \qquad \ldots(24)$$

whose solution is

$$\bar{\phi}_{1+}(p, z) - \bar{\phi}_{1+}(-p, z) = A(p) e^{-Y_1 z} + B(p) e^{Y_1 z}. \qquad \ldots(25)$$

Since $\bar{\phi}_{1+}(p, z) - \bar{\phi}_{1+}(-p, z)$ is bounded when z tends to infinity, it follows that $B(p) = 0$ and

$$\bar{\phi}_{1+}(p, z) - \bar{\phi}_{1+}(-p, z) = A(p) e^{-Y_1 z}. \qquad \ldots(26)$$

Similarly

$$\bar{\psi}_{1+}(p, z) - \bar{\psi}_{1+}(-p, z) = D(p) e^{-\delta_1 z}, \quad \delta_1 = \pm (p^2 - k''^2)^{1/2}. \qquad \ldots(27)$$

The criteria for signs of Y_1 and δ_1 are the same as for Y. Putting $z = h$ in (26) and in its derivative with respect to z and eliminating $A(p)$, we have

$$\bar{\phi}_{1+}(p) - \bar{\phi}_{1+}(-p) = \frac{-1}{Y_1} [\bar{\phi}'_{1+}(p) - \bar{\phi}'_{1+}(-p)]. \qquad \ldots(28)$$

5. Solution of the Wiener-Hopf Type Equation

The functional equation (28) is solved by involving the Wiener-Hopf technique. We write (28) as

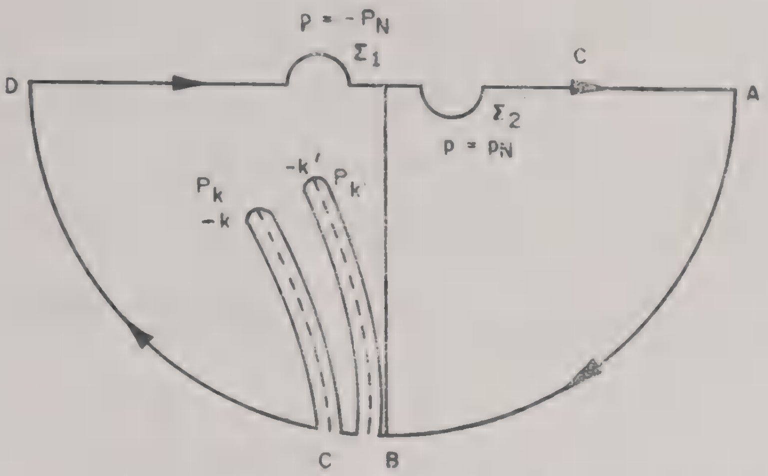

FIG. 2 Contour of integration in lower half of the complex plane.

$$\bar{\phi}_{1+}(p) + \frac{1}{(p-k')^{1/2}}\left[\frac{\bar{\phi}'_{1+}(p)}{(p+k')^{1/2}} - \frac{\bar{\phi}'_{1+}(k')}{(2k')^{1/2}}\right] - \frac{\bar{\phi}_{1+}(k')}{(-2k'(p+k')^{1/2})}$$

$$= \bar{\phi}_{1+}(-p) + \frac{1}{(p+k')^{1/2}}\left[\frac{\bar{\phi}'_{1+}(-p)}{(p-k')^{1/2}} - \frac{\bar{\phi}'_{1+}(k')}{(-2k')^{1/2}}\right]$$

$$- \frac{\bar{\phi}'_{1+}(k')}{(2k'(p-k'))^{1/2}}. \qquad \ldots(29)$$

The left hand member of (29) is analytic in the region $\beta > -k_2$ and the right hand member in $\beta < k_2$. By analytic continuation, they represent an entire function. Each member tends to zero in its respective region of analyticity as $|p| \to \infty$. By an extension of Liouville theorem (Sec. 2.61, Titchmarsh[8]) the entire function is identically zero. Thus we have

$$\bar{\phi}_{1+}(p) = -\frac{1}{Y_1}\bar{\phi}'_{1+}(p) + \bar{\phi}'_{1+}(k')\, g(p) \qquad \ldots(30)$$

where

$$g(p) = \frac{1}{[2k'(p-k')]^{1/2}} + \frac{1}{[(-2k')(p+k')]^{1/2}}. \qquad \ldots(31)$$

Similarly

$$\bar{\psi}_{1+}(p) = \frac{-1}{\delta_1}\bar{\psi}'_{1+}(p) + \bar{\psi}'_{1+}(k'')\, h(p) \qquad \ldots(32)$$

where $h(p)$ is obtained from $g(p)$ by replacing k' by k''.

Using (30) and (32) in the conditions (21-23), eliminating $\bar{\phi}'_{1+}(p)$ and $\bar{\psi}'_{1+}(p)$ between the resulting equations, $\bar{\phi}_+(p) + \bar{\phi}_+(-p)$ is obtained in terms $\bar{\phi}'_+(p) + \bar{\phi}'_+(-p)$. The result is used in (20) to obtain

$$\bar{\phi}'_+(p) + \bar{\phi}'_+(-p) = -\frac{(2ik \sin\theta \cos\theta)\, D \cos(kh \sin\theta)}{p^2 - k^2 \cos^2\theta} + \frac{2g(p)(-\lambda_1 k_1^2 + \mu_1 Y_1^2)\,\bar{\phi}'_{1+}(k')}{\lambda k^2 f(p)} \quad \dots(33)$$

where

$$f(p) = \frac{\tanh Yh}{Y} + \frac{(2p^2 - k''^2)(-\lambda_1 k'^2 + 2\mu_1 Y_1^2)}{\lambda k^2 Y_1 k''^2} - \frac{4p^2 \mu_1 \delta_1}{\lambda k^2 k''^2} \dots (34)$$

From (19) and (33), it follows that

$$\bar{\phi}_+(p, z) + \bar{\phi}_+(-p, z) = \frac{\sinh Yz}{Y \cosh Yh}\left[\frac{2g(p)(-\lambda_1 k'^2 + 2\mu_1 Y_1^2)\,\bar{\phi}'_{1+}(k')}{\lambda k^2 f(p)}\right] - \frac{(2ik\, D \cos\theta) \sin(kz \sin\theta)}{p^2 \quad k^2 \cos^2\theta}. \quad \dots(35)$$

The potential $\phi(x, z)$ is given by the inverse Fourier transform, i.e.,

$$\phi(x, z) = \frac{1}{2\pi}\int_{-\infty + i\beta}^{\infty + i\beta} \bar{\phi}_+(p, z)\, e^{-ipx}\, dp, \; x \geqslant 0 \quad \dots(36)$$

where $-k_2 < \beta < k_2$.

6. Reflected Waves.

The factor $\exp(-ipx) = \exp(-i\alpha x)\exp(\beta x)$ in (36) makes the integral vanish $\beta \to -\infty$ in the lower part of the complex plane if $x \geqslant 0$. For waves in region $x \geqslant 0$, the contour is in the lower half of the complex plane where $\bar{\phi}_+(-p, z)$ is analytic and hence

$$\phi(x, z) = \frac{1}{2\pi}\int_{-\infty + i\beta}^{\infty + i\beta} [\bar{\phi}_+(p, z) + \bar{\phi}_+(-p, z)]\, e^{-ipx}\, dp, \; x \geqslant 0. \quad \dots(37)$$

The contour of integration excludes the poles $p = \pm k \cos\theta$ and includes the poles $p = \pm p_n$, $(n = 1, 2, 3, \dots)$ where $\pm p_n$ are the roots of the period equation $f(p) = 0$.

Schermann[9] has shown that the period equation has a finite number of real roots $p = \pm p_n$, n denoting the nth normal mode of the generalised Rayleigh waves. The conditions Re $(Y) = 0$, Re $(Y_1) = 0$ and Re $(\delta_1) = 0$ have been discussed by Ewing *et al.*[10] to show that the branch cuts are hyperbolic. Im (Y), Im (Y_1) and Im (δ_1) change signs along the two sides of the branch cuts as indicated in Fig. 2. The line integral (37) is taken along the real axis ($\beta = 0$) of the complex plane. Applying Cauchy residue theorem to the closed contour results in

$$\int_{-\infty}^{\infty} + \int_{A}^{B} + \int_{p_{k_l}} + \int_{p_{k_u}} + \int_{p_k} + \int_{C}^{D} = 2\pi i \,(\text{sum of residues}) \qquad \ldots(38)$$

where

$$\int_{-\infty}^{\infty} = \underset{\epsilon_1 \to 0}{\text{limit}} \int_{-\infty}^{-p_n - \epsilon_1} + \int_{\Sigma_1} + \underset{\epsilon_1, \epsilon_2 \to 0}{\text{limit}} \int_{-p_n + \epsilon_1}^{-k \cos \theta - \epsilon_2} + \int_{\Sigma_2} + \underset{\epsilon_2 \to 0}{\text{limit}} \int_{-k \cos \theta + \epsilon_2}^{k \cos \theta - \epsilon_2}$$

$$+ \int_{\Sigma_3} + \underset{\epsilon_1, \epsilon_2 \to 0}{\text{limit}} \int_{k \cos \theta + \epsilon_2}^{p_n - \epsilon_1} + \int_{\Sigma_4} + \underset{\epsilon_1 \to 0}{\text{limit}} \int_{p_n + \epsilon_1}^{\infty} \qquad \ldots(39)$$

ϵ_1 and ϵ_2 being the radii of circular indentations Σ_1, Σ_2, Σ_3 and Σ_4 around $p = -p_n$, $p = p_n$, $p = -k \cos \theta$ and $p = k \cos \theta$ respectively. The contribution due to indentations at $p = \pm k \cos \theta$ are

$$-\tfrac{1}{2} D \sin (kz \sin \theta) \exp (-ik\, x \cos \theta) \qquad \ldots(40)$$

and

$$\tfrac{1}{2} D \sin (kz \sin \theta) \exp (ikx \cos \theta). \qquad \ldots(41)$$

And the contribution due to indentations at $p = \pm p_n$ are

$$-\sum_n \frac{i(-\lambda_1 k'^2 + 2\mu_1 Y_{1n}^2)\, g(p_n)\, \bar{\phi}'_{1+}(k') \sin Y_n z\, e^{-ip_n x}}{Y_n \lambda k^2 f'(p_n) \cos Y_n h},$$

$$Y_{1n} = [(p_n^2 - k'^2)]^{1/2} \qquad \ldots(42)$$

and

$$\sum_n \frac{i(-\lambda_1 k'^2 + 2\mu_1 Y_{1n}^2)\, g(-p_n)\, \bar{\phi}_{1+}(k') \sin Y_n z\, e^{ip_n x}}{\lambda Y_n k^2 f'(-p_n) \cos Y_n h},$$

$$Y_n = [(k^2 - p_n^2)]^{1/2} \qquad \ldots(43)$$

In (40), the wave is half the incident wave whereas (41) gives the wave reflected from the rigid plane boundary and then from the free surface of the ocean. In (42) and (43), we have the generalized Rayleigh waves multiply reflected from the boundaries of the ocean.

7. Scattered Waves

Let us evaluate the integral along the branch cut P_k. On P_k, Re $(Y) = 0$ and Im (Y) changes signs along two sides of the branch cut. It is found that the integrals along two sides cancel each other.

We now consider

$$I(x, z) = \frac{1}{2\pi} \int_{p_k'} [\bar{\phi}_+(p, z) + \bar{\phi}_+(-p, z)]\, e^{-ipx}\, dp. \qquad \ldots(44)$$

On p_k', we use $p = -k' - iu$ such that $Y_1 = \pm i\,(2k_2'\,u)^{1/2} = \pm i\bar{Y}_1$, $k_1' = 0$. Integrating (44) along two sides of p_k, it follows that

$$I(x, z) = \frac{i}{2\pi} \int_{p_k'} [[\bar{\phi}_+(p, z) + \bar{\phi}_+(-p, z)]_{Y_1 = i\bar{Y}_1} - [\bar{\phi}_+(p, z)$$

$$+ \bar{\phi}_+(-p, z)]_{Y_1 = -i\bar{Y}_1}]\, e^{-k_2' x}\, e^{-ux}\, du. \qquad \ldots(45)$$

This is evaluated by using the result (Ewing *et al.*[10], p. 52)

$$\int_0^\infty \sqrt{u}\, G(u)\, e^{-ux}\, du = \frac{G(0)\,\Gamma(3/2)}{x^{3/2}} + \frac{G'(0)\,\Gamma(5/2)}{x^{5/2}} + \ldots \qquad \ldots(46)$$

where $\Gamma(x)$ is the Gamma function. The first term of (45) is obtained to be

$$I(x, z) = C x^{-3/2 - k_2' x} \sin((k^2 + k_2'^2)^{1/2} z) \qquad \ldots(47)$$

where

$$C = \frac{2k''^2 H_1(0)}{\pi H_2(0)} \Big[-\frac{\lambda_1 k^2 e^{i\pi/4} (k''^2 - 2k'^2)(k^2 + k_2'^2)^{1/2}}{\sqrt{-k'}}$$

$$+ \lambda k^2 k''^2 (2k_2')^{1/2} \tan((k^2 + k_2'^2)^{1/2} h)$$

$$- 4i\,\mu_1 k'^2 (k_2'^2 + k''^2)^{1/2} (k^2 + k_2'^2)^{1/2} (2k_2')^{1/2} \Big] \qquad \ldots(48)$$

$$H_1(0) = \lambda_1 k^2\, \Gamma(3/2)\, \bar{\phi}'_{1+}(k') \qquad \ldots(49)$$

$$H_2(0) = (\lambda_1^2 k'^4)(k^2 + k_2'^2)(k''^2 - 2k'^2)^2 \cos((k^2 + k_2'^2)^{1/2} h). \qquad \ldots(50)$$

The scattered wave in (47) has the behaviour of a cylindrical wave close to the scatterer and a plane wave at distant points.

8. Conclusions

The potential for the incident, reflected, Rayleigh and scattered waves in (3), (40)-(43) and (47) vanishes on the free surface ($z = 0$) of the ocean and the boundary condition (6) is satisfied. It can be further established that the condition (7) and all other boundary conditions are satisfied by the potentials of various waves. The horizontal component u of the displacement excited by various waves is absent on the free surface of the ocean. Thus the particle motion in the surface of the ocean is at right angles to it and the horizontal component of the displacement is absent. The waves reflected from the rigid plane boundary and then from the surface of the ocean are obtained in (41) having exactly half the amplitude of the incident wave. Multiply reflected generalised Rayeigh waves in (42) - (43) and the scattered waves in (47) are plane waves. When z is small, the scattered waves are of the form of exp $(-k'_2 r)/\sqrt{r}, r$ being the distance from the corner of the plane boundary. Near the scatterered, the near-field behaves as a cylindrical wave. The scatted waves propagate with the speeds of the waves in the solid and not with the speed of the waves in the ocean.

Numerical computations have been worked out for the scattered field and the generalised Rayleigh waves in the oceanic layer. For a basaltic ocean bottom,

$$\rho_2 = 3\rho_1,\ \alpha_2 = \sqrt{3}\beta_2,\ \beta_2 = 3\alpha_1 \text{ and } h = 5.7\text{km}.$$

Amplitude of the scattered wave is plotted versus the distance from the scatterer just below the free surface. It falls steadily as the waves proceed away from the scatterer (Fig. 3). The displacement excited by the scattered waves in the free surface falls rapidly

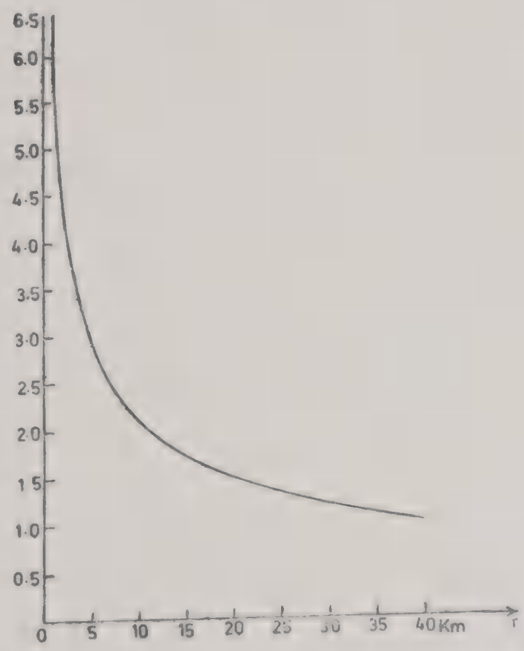

Fig. 3. Amplitude of the scattered wave just below the surface.

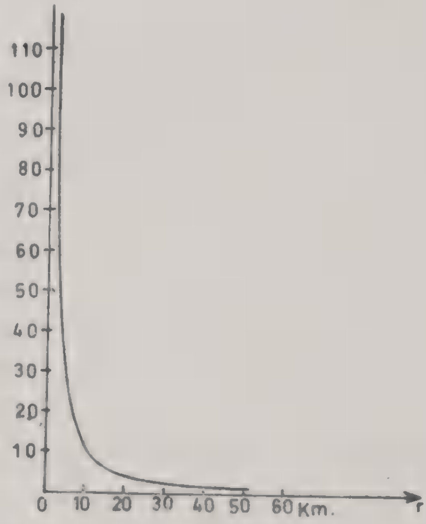

FIG. 4. Amplitude of the displacement excited by scattered waves in the free surface.

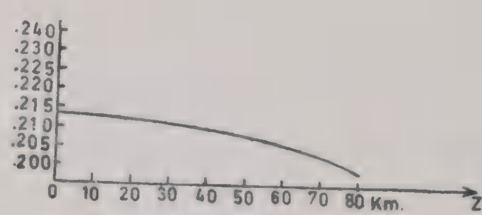

FIG. 5. Amplitude of the displacement excited by Rayleigh waves versus depth.

upto a distance of 10 km and then receds slowly when $\alpha_1 = 1.52$ km/sec, $\alpha_2 = 7.95$ km/sec. and $\beta_2 = 4.56$ km/sec. (Fig. 4). The amplitude of the displacement excited by Rayleigh waves is computed in terms of depth below the free surface and is found to fall down very slowly (Fig. 5).

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A NOTE ON *S*-CLOSED SPACES

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The class of s-closed spaces has been recently introduced by Di Maio and Noiri [*Indian J. pure appl. Math.*, 18 (1987), 226-33]. In this paper it is pointed out that among weakly-T_2 spaces the class of s-closed spaces coincides with the class of S-closed spaces due to Thompson. We show that every infinite topological space is embeddable as a closed subspace in a connected S-closed space which is not s-closed.

In a recent paper Di Maio and Noiri[3] have introduced and studied the class of s-closed spaces. They pointed out that s-closedness is equivalent to cd-compactness due to Carnahan[2] and to weak RS-compactness due to Hong[6]. The purpose of this note is to examine the relationship between the class of s-closed spaces and the more familiar class of S-closed spaces which was introduced by Thompson[9] and studied by Cameron[1]. It is clear that every s-closed space is S-closed, but is the converse true ? No example of an S-closed space which is not s-closed has been provided in Di Maio and Noiri[3], and according to Di Maio[4] this is an open question. In this paper we observe that these two classes of spaces coincide for a very large class of spaces, namely the weakly-T_2 spaces. We show that every infinite topological space can be represented as a closed subspace of a connected S-closed space which is not s-closed. We also prove that there exist semi-T_2 S-closed spaces which fail to be s-closed.

Let S be a subset of a space X. We denote the closure and the interior of S with respect to the space X by $\text{cl}_X\, S$ and $\text{int}_X\, S$ respectively. A subset S of X is called semi-open [resp. regular closed] if $S \subset \text{cl}_X\,(\text{int}_X\, S)$ [resp. $S = \text{cl}_X\,(\text{int}_X\, S)$]. In particular, every regular closed set is semi-open and the closure of every semi-open set is regular closed. The complement of a semi-open set is called a semi-closed set. The

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semi-closure of a subset S of X is the smallest semi-closed set containing S and is denoted by $\mathrm{scl}_X\, S$. One easily verifies that $\mathrm{scl}_X\, S = S \cup \mathrm{int}_X\,(\mathrm{cl}_X\, S)$ for any subset S of X.

Definition—A space X is called S-closed Thompson[9] [resp. s-closed[3]] if every semi-open cover of X contains a finite subfamily the closures [resp. the semi-closures] of whose members cover X.

It is obvious that every s-closed space is S-closed. Recall that a space X is said to be extremally disconnected if $\mathrm{cl}_X\, U$ is open for each open subset U of X. It is observed in Maio and Noiri[3] that if X is exteremally disconnected then $\mathrm{cl}_X\, S = \mathrm{scl}_X\, S$ for every semi-open set S in X. Thompson[9] (Theorem 7) has proved that every Hausdorff S-closed space is extremally disconnected. Herrmann[5] (Theorem 3.7) has extended this result to the more general class of weakly-T_2 spaces Soundarajan[8], i.e. spaces in which every singleton is the intersection of regular closed sets. Hence, among weakly-T_2 spaces the class of S-closed spaces coincides with the class of s-closed spaces.

The question whether there is any difference between S-closed spaces and s-closed spaces is now settled by the following results.

Theorem—Every infinite topological space Y can be represented as a closed subspace of a space X which is s-closed.

PROOF : Let Z be an infinite T_1 space and let $Z_1 = Z \times \{1\}$ and $Z_2 = Z \times \{2\}$. We may assume that $Y \cap (Z_1 \cup Z_2)$ is empty. Let $X = Y \cup Z_1 \cup Z_2$. For $i = 1, 2$ let $\mathscr{B}_i = \{W_i \subset Z_i : W_i = V \times \{i\}$ for some open subset V of $Z\}$. Let $\mathscr{B}_3 = \{G \subset X : G = U \cup C_1 \cup C_2$ where $U \subset Y$ is open in Y and C_1 resp. C_2 are cofinite subsets of Z_1 resp. $Z_2\}$. It is easily checked that a topology on X is defined by taking $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ as a base. Clearly Z_1 and Z_2 are open in X and Y is a closed subspace of X. For $i = 1, 2$. we obviously have $\mathrm{cl}_X\, Z_i = Y \cup Z_i$ and $\mathrm{scl}_X\, Z_i = Z_i$. For each $y \in Y$, if $S_y = Z_1 \cup \{y\}$ then S_y is semi-open in X and $\mathrm{scl}_X\, S_y = S_y$. Hence $\{S_y : y \in Y\} \cup \{Z_2\}$ is a semi-open cover of X which has no finite subfamily the semi-closures of whose members cover X. Thus X is not s-closed.

Corollary 1—There exist S-closed spaces which are not s-closed.

PROOF : Let Z be an infinite S-closed T_1 space. Since $Z_1 \cup Z_2$ is dense in X, by Lemma 2.2 and Theorem 3.4 in Noiri[7] it follows that X is S-closed. In particular, if Z is an infinite set cainrying the cofinite topology then X is even a connected S-closed T_1 space which fails to be s-closed.

Recall that a space is said to be semi-T_2 if every pair of distinct points can be separated by disjoint semi-open sets.

Corollary 2—There exist semi-T_2 S-closed spaces which are not s-closed.

PROOF : By our theorem and Corollary 1, if Y is an infinite discrete space and Z is an infinite Hausdorff S-closed space then X is T_1 and S-closed but not s-closed. It

is easily checked that X is semi-T_2. For example, if $y_1, y_2 \in Y$ and $y_1 \neq y_2$ then $Z_1 \cup \{y_1\}$ and $Z_2 \cup \{y_2\}$ are disjoint semi-open sets containing y_1 and y_2 respectively. The other cases are treated in a similar fashion.

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